Cluster expansions: Overview and new convergence results

I. General set-up, main examples and basic expressions

Roberto Fernández
CNRS - Université de Rouen

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Part I

General set-up and main examples

Most of the course will deal with “hard-core” polymers (also called *level 3*)

The set-up is more general that what it seems

Starting point: basic issue, graph-theoretical set-up, major examples

Outline

The basic (Level 3) setup

Goal: To study systems of objects constrained only by a "non-overlapping" condition

Countable family $\mathcal{P}$ of objects: polymers, animals, ..., characterized by

- An *incompatibility* constraint:

$$\begin{array}{ccc} \gamma \nsim \gamma' & & \text{incompatible} \\ & \text{if } \gamma, \gamma' \in \mathcal{P} & \\ \gamma \sim \gamma' & & \text{compatible} \end{array}$$

  For simplicity: each polymer incompatible with itself ($\gamma \nsim \gamma$, $\forall \gamma \in \mathcal{P}$)

- A family of *activities* $\boldsymbol{z} = \{z_\gamma\}_{\gamma \in \mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$.

The basic ("finite-volume") measures

Defined, for each *finite* family $\mathcal{P}_\Lambda \subset \mathcal{P}$, by weights

$$\mathrm{W}_\Lambda\big(\{\gamma_1,\gamma_2,\ldots,\gamma_n\}\big) \;=\; \frac{1}{\Xi_\Lambda(\boldsymbol{z})}\, z_{\gamma_1} z_{\gamma_2}\cdots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

for $n \geq 1$ $\gamma_1,\gamma_2,\ldots,\gamma_n \in \mathcal{P}_\Lambda$, and $\mathrm{W}_\Lambda(\emptyset) = 1/\Xi_\Lambda$, where

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} z_{\gamma_1} z_{\gamma_2}\ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

- Λ = some label, often finite subset of a countable set
- As compatible polymers are necessarily different,

$$\frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} [\bullet] \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}} \;=\; \sum_{\{\gamma_1,\ldots,\gamma_n\}\subset\mathcal{P}_\Lambda} [\bullet] \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

(different situation below for cluster expansion)

The questions:

- Existence of the limit $\mathcal{P}_\Lambda \to \mathcal{P}$ ("thermodynamic limit")
- Properties of the resulting measure (mixing properties, dependency on parameters,...)
- Asymptotic behavior of Ξ_Λ

Motivation

Immediate:

- *Physics:* Grand-canonical ensemble of polymer gas with activities z_γ and hard-core interaction
- *Statistics:* Invariant measure of point processes with not-overlapping grains and birth rates z_γ

Less immediate:

- Statistical mechanical models at high and low temperatures are mapped into such systems
- More generally: most perturbative arguments in physics involve maps of this type (choice of the "right" variables)
- Zeros of the partition functions Ξ_Λ (phase transitions, chromatic polynomials, Lovász lemma)

Graph-theoretical framework

Equivalently, consider the *incompatibility graph* $\mathcal{G} = (\mathcal{P}, \mathcal{E})$

Unoriented graph with:

- Vertices = polymers
- Edges = incompatible pairs

$$\gamma \nsim \gamma' \quad \text{iff} \quad \{\gamma, \gamma'\} \in \mathcal{E} \quad \text{or} \quad \gamma \leftrightarrow \gamma' \tag{1}$$

  (contrast!)

- $\mathcal{E}$ is arbitrary; vertices can be of infinite degree (polymers incompatible with infinitely many other polymers)

WARNING! There will be other graphs (up to three levels)

Polymers as lattice gases

In this graph-theoretical framework:

- Incompatible polymers = neighboring vertices
- Polymer system = hard-core gas in a complicated lattice
- *Neighborhood of* γ_0:

$$\begin{aligned} \mathcal{N}^*_{\gamma_0} &= \{\gamma \in \mathcal{P} : \gamma \nsim \gamma_0\} \\ \mathcal{N}_{\gamma_0} &= \mathcal{N}^*_{\gamma_0} \setminus \{\gamma_0\} \end{aligned}$$

- *Independent vertices* = non-neighboring vertices
- *Independent sets* = sets formed by independent vertices

Thus,

$$\Xi_\Lambda(\boldsymbol{z}) = \sum_{\substack{\Gamma \subset \mathcal{P}_\Lambda \\ \text{independent}}} \boldsymbol{z}^\Gamma \qquad \text{with} \quad \boldsymbol{z}^\Gamma = \prod_{\gamma \in \Gamma} z_\gamma$$

Example: Single-call loss networks

Definition

- $\mathcal{P}$ = finite subsets of $\mathbb{Z}^d$ —the *calls*
- A call γ is attempted with Poissonian rates z_γ
- Call succeeds if it does not intercept existing calls
- Once established, calls have an exp(1) life span

Remarks

- Basic measures are invariant for the finite-region process $(\gamma \nsim \gamma' \Longleftrightarrow \gamma \cap \gamma' \neq \emptyset)$
- Thermodynamic limit: infinite-volume process
- Discrete point process with hard-core conditions

Statistical mechanical lattice models

Their ingredients are:

- *Lattice* $\mathbb{L}$ countable set of sites (e.g. $\mathbb{Z}^d$)
- *Single-site space* $(E, \mathcal{F}, \mu_E)$ with natural measure structure (e.g. counting measure if E countable, Borel if $E \subset \mathbb{R}^d$)
- *Configuration space* $\Omega = E^{\mathbb{L}}$, with product measure
- *Interaction* $\Phi = \{\phi_B : B \subset\subset \mathbb{L}\}$ where $\phi_B = \phi_B(\omega_B)$
    - *Bonds* are sets B such that $\phi_B \neq 0$
    - *Exclusions:*
        - $\Phi_B(\omega_B) = \infty$ (physicist)
        - $\Omega_{\text{allowed}} \subset \Omega$ (math-phys)
    - *Two body:* $\phi_B = 0$ unless $B = \{x, y\}$

Statistical mechanical measures

Their finite-volume versions are defined by

- *Hamiltonians:* For $\Lambda \subset\subset \mathbb{L}$, and boundary condition σ

$$H_\Lambda(\omega \mid \sigma) = \sum_{B \subset \Lambda} \phi_B(\omega_\Lambda \sigma)$$

- Boltzmann Probability densities (weights)

$$W_\Lambda(\omega \mid \sigma) \;=\; \frac{\exp\{-\beta H_\Lambda(\omega \mid \sigma)\}}{Z_\Lambda^\sigma}$$

$(\omega, \sigma \in \Omega_{\mathrm{all}})$ with

$$Z_\Lambda^\sigma \;=\; \int_{\Omega_{\mathrm{all}}} \exp\{-\beta H_\Lambda(\omega \mid \sigma)\} \bigotimes_{x \in \Lambda} \mu_E(d\omega_x)$$

(β = inverse temperature)

Warning on notation

- Often β is absorbed:

$$\beta\phi_B \;\to\; \phi_B \quad , \quad \beta H_\Lambda \to H_\Lambda$$

- Also, single site terms $\phi_{\{x\}}(\omega_x)$ can be absorbed in μ_E

$$\mu_E(d\omega_x) \;\to\; \mu_x(d\omega_x) = \mathrm{e}^{-\beta\phi_{\{x\}}(\omega_x)}\mu_E(d\omega_x)$$

Example zero: Hard-core lattice gases

$\mathbb{L}$ =vertices of a graph (eg. $\mathbb{Z}^d$), $E = \{0, 1\}$
($\mathcal{F}$ =discrete, μ_E =counting)

$$\phi_B(\omega) \;=\; \begin{cases} -u\,\omega_x & \text{if } B = \{x\} \\ \infty & \text{if } B = \{x, y\} \text{ n.n.} \\ 0 & \text{otherwise} \end{cases}$$

Let

$$\Gamma(\omega) = \{x : \omega_x = 1\}$$

Then, for $\Lambda \subset\subset \mathbb{L}$,

$$W_\Lambda(\omega \mid 0) \;=\; \frac{1}{Z_\Lambda^0} \prod_{x \in \Gamma(\omega_\Lambda)} \mathrm{e}^{\beta u} \prod_{x,y \in \Gamma(\omega_\Lambda)} \mathbb{1}_{\{x \not\leftrightarrow y\}}$$

Lattice gas = polymer model

This is a polymer model with

- $\mathcal{P} = \{\text{vertices of } \mathbb{L}\}$
- $x \not\sim y$ iff x and y are graph neighbors
- $z_x = \mathrm{e}^{\beta u}$

(For Sokal-like people *all* polymer models are of this type)

Ising model at low temperatures

$\mathbb{L} = \mathbb{Z}^d$, $E = \{-1, 1\}$, ($\mathcal{F}$ =discrete, μ_E =counting)

$$\phi_B(\omega) = \begin{cases} -J\,\omega_x\omega_y & \text{if } B = \{x, y\} \text{ n.n.} \\ 0 & \text{otherwise} \end{cases}$$

Write $-J\,\omega_x\omega_y = -J\,(\omega_x\omega_y - 1) - J$

Call a bond $B = \{x, y\}$ *excited* or *frustrated* if $\omega_x\omega_y = -1$:

$$H_\Lambda(\omega \mid +) = 2J\,F_\Lambda(\omega) - J N_\Lambda\ ;$$

$$\begin{aligned} F_\Lambda(\omega) &= \#\{B \text{ frustrated} : B \cap \Lambda \neq \emptyset\} \\ N_\Lambda &= \#\{B : B \cap \Lambda \neq \emptyset\} \end{aligned}$$

As N_Λ is independent of ω

$$W_\Lambda(\omega \mid +) = \frac{\exp\{-2\beta J\,F_\Lambda(\omega)\}}{\sum_{\sigma_\Lambda} \exp\{-2\beta J\,F_\Lambda(\sigma)\}}$$

Contour representation

- Place a plaquette (segment) orthogonally at the midpoint of each frustrated bond
- These plaquettes form a family of disjoint closed connected surfaces (curves)
- Each such closed surface is a *contour*. Denote

$$\mathcal{C}_\Lambda \;=\; \{\text{contours } \gamma : \gamma \subset \Lambda\}$$

- Contours are disjoint: $\gamma \sim \gamma' \iff \gamma \cap \gamma' = \emptyset$
- Each ω is in one-to-one correspondence with a *compatible* family of contours $\Gamma(\omega)$

Contour polymer model

$$\begin{aligned}\exp\{-2\beta J\, F_\Lambda(\omega)\} &= \exp\Big\{-\sum_{\gamma\in\Gamma(\omega)} 2\beta J\; |\gamma|\Big\}\\ &= \prod_{\gamma\in\Gamma(\omega)} z_\gamma\end{aligned}$$

with $z_\gamma = \exp\{-2\beta J\; |\gamma|\}$. Hence

$$W_\Lambda(\omega \mid +) \;=\; \frac{1}{\Xi_\Lambda} \prod_{\gamma\in\Gamma(\omega)} z_\gamma$$

with

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{C}_\Lambda^n} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

Generalization: LTE for Ising ferromagnets

$\mathbb{L}$ =any, $E = \{-1, 1\}$, interactions

$$\phi_B(\omega) \;=\; -J_B\,\omega^B \quad , \text{with } J_B \geq 0$$

$[\omega^B := \prod_{x\in B} \omega_x]$. Without loss, free boundary conditions:

$$H_\Lambda(\omega) \;=\; -\sum_{B\in\mathcal{B}_\Lambda} J_B\,\omega^B$$

with

$$\mathcal{B}_\Lambda \;=\; \Big\{B : J_B > 0 \text{ and } B \subset \Lambda\Big\}$$

[for $H_\Lambda(\cdot \mid +)$ use $\mathcal{B}_\Lambda^+$, etc]

Generalized contours

Write

$$\begin{aligned} H_\Lambda(\omega) &= -\sum_{B\in\mathcal{B}_\Lambda} J_B\big(\omega^B - 1 + 1\big) \\ &= -\sum_{B\in\mathcal{B}_\Lambda} J_B\big(\omega^B - 1\big) - \sum_{B\in\mathcal{B}_\Lambda} J_B \end{aligned}$$

- A bond B is *excited* or *frustrated* if $\omega^B = -1$
- $\Gamma(\omega_\Lambda)$ = set of frustrated bonds in Λ
- A *contour* is a maximal connected component of Γ (connexion = intersection)
- $\mathcal{C}_\Lambda$ = set of possible contours in Λ

Contours and probability weights

$$W_\Lambda(\omega) \;=\; \frac{\prod_{\gamma\in\Gamma(\omega_\Lambda)} \mathrm{e}^{-\beta E(\gamma)}}{\widetilde{Z}_\Lambda}$$

where $E(\gamma) = \sum_{B\in\gamma} 2J_B$ and

$$\widetilde{Z}_\Lambda \;=\; \sum_{\sigma_\Lambda} \prod_{\gamma\in\Gamma(\sigma_\Lambda)} \mathrm{e}^{-\beta E(\gamma)} \;=\; \sum_{\Gamma\in\mathcal{C}_\Lambda} N_\Gamma \prod_{\gamma\in\Gamma} \mathrm{e}^{-\beta E(\gamma)}$$

with $N_\Gamma \;=\; \{\omega_\Lambda : \Gamma(\omega_\Lambda) = \Gamma\}$

We compute N_Γ with a little help from group theory

Contours and group theory

- $\Gamma(\omega_\Lambda) = \Gamma(\sigma_\Lambda)$ iff $\omega^B = \sigma^B$ for all $B \in \mathcal{B}_\Lambda$
- $\Gamma(\omega_\Lambda) = \Gamma(\sigma_\Lambda)$ iff $(\omega \cdot \sigma)^B = 1$ for all $B \in \mathcal{B}_\Lambda$, where

$$(\omega \cdot \sigma)_x = \omega_x \sigma_x$$

  Site-wise product

- $\Gamma(\omega_\Lambda) = \Gamma(\sigma_\Lambda)$ iff $\omega = \chi \cdot \sigma$ for some $\chi \in \mathcal{S}_\Lambda$ with

$$\mathcal{S}_\Lambda \;=\; \left\{\chi : \chi^B = 1 \text{ for all } B \in \mathcal{B}_\Lambda\right\}$$

  Symmetry group

- $N_\Lambda = |\mathcal{S}_\Lambda|$

Ferromagnetic LT polymer model

Finally,

$$Z_\Lambda = |\mathcal{S}_\Lambda| \; \Xi_\Lambda^{\rm LT}$$

with

$$\Xi_\Lambda^{\rm LT}(\boldsymbol{z}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{C}_\Lambda^n} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

for

$$z_\gamma \;=\; \exp\Bigl\{-2\beta \sum_{B\in\gamma} J_B\Bigr\}$$

($|z_\gamma|$ small for β large) and

$$\gamma \sim \gamma' \quad\Longleftrightarrow\quad \gamma\cap\gamma' = \emptyset$$

Geometrical polymer models

Polymers of previous examples (loss networks, low-T contours) are points of a set

These are the original polymer models of Gruber and Kunz

Formally, a geometrical polymer model is defined by:

- A set $\mathbb{V}$ (eg. possible calls, surfaces)
- A family $\mathcal{P}$ of finite subsets of $\mathbb{V}$ (eg. connected)
- Activity values $(z_\gamma)_{\gamma \in \mathcal{P}}$
- The relation $\gamma \sim \gamma' \iff \gamma \cap \gamma' = \emptyset$

In this case $\mathcal{P}_\Lambda = \{\gamma \in \mathcal{P} : \gamma \subset \Lambda\}$, $\Lambda \subset\subset \mathbb{V}$

General geometrical polymers

Vertex-set polymers

$\mathbb{V}$ = vertex set of a graph (lattice, dual lattice)

- Polymers are defined through connectivity properties (graph-connected)
- Compatibility determined by graph distances (overlapping, being neighbors or sufficiently close)

WARNING! Second-level graph. On top: incompatibility graph

Decorated geometrical polymers

$\gamma = (\underline{\gamma}, D_\gamma)$ where

- $\underline{\gamma}$ = finite subset of $\mathbb{V}$ ("base")
- D_γ some additional attribute (color, "decoration")
- In this case: $x \in \gamma$ means $x \in \underline{\gamma}$, etc

Part II

Expanding the log of partition functions

Let us spend some time discussing

- Why issues reduce to the study partition functions
- Information yielded by expansions of logs of part. functions

We leave for later the convergence problem.

Outline

Ratios of partition functions

Partition functions play a central role. Three reasons:

- Correlations are ratios of partition functions
- So are characteristic and moment-generating functions
- (Complex) zeros of partition functions related to phase transitions, coloring problems, etc

Polymer correlation functions

Let

- Prob_Λ the basic measure in $\mathcal{P}_\Lambda$
- $\gamma_1, \ldots, \gamma_k$ mutually compatible polymers in $\mathcal{P}_\Lambda$

Then

$$\mathrm{Prob}_\Lambda\big(\{\gamma_1, \ldots, \gamma_k \text{ are present}\}\big) \;=\; z_{\gamma_1} \cdots z_{\gamma_k} \, \frac{\Xi_{\Lambda\setminus\{\gamma_1,\ldots,\gamma_k\}^*}}{\Xi_\Lambda}$$

where

$$\Xi_{\Lambda\setminus\{\gamma_1,\ldots,\gamma_k\}^*} = \text{partition function of polymers in } \mathcal{P}_\Lambda \text{ compatible with } \gamma_1, \ldots, \gamma_k$$

Statistical mechanical correlations

Likewise, for the stat-mech models, let

- $\mathrm{Prob}_\Lambda(\,\cdot \mid \sigma)$ be the measure in Λ with b.c. σ
- A_Δ be an event depending only on spins in $\Delta \subset \Lambda$

Then

$$\mathrm{Prob}_\Lambda(A_\Delta \mid \sigma) = \int \mathbb{1}_{\{A_\Delta\}}(\omega_\Delta)\, \mathrm{e}^{-\beta H_\Delta(\omega_\Delta)}\, \frac{Z_{\Lambda\setminus\Delta}^{\omega_\Delta\,\sigma_{\mathbb{L}\setminus\Lambda}}}{Z_\Lambda^\sigma} \bigotimes_{x\in\Delta} \mu_E(d\omega_x)$$

where

$$Z_{\Lambda\setminus\Delta}^{\omega_\Delta\,\sigma_{\mathbb{L}\setminus\Lambda}} = \textit{partition function in } \Lambda\setminus\Delta \textit{ with condition } \omega \textit{ in } \Delta \textit{ and } \sigma \textit{ outside } \Lambda$$

Characteristic/moment-generating functions

Let $\alpha : \mathcal{P} \to \mathbb{R}$ and

$$S_\Lambda(\gamma_1, \ldots, \gamma_n) \;=\; \sum_{i=1}^{n} \alpha(\gamma_i)$$

for $\{\gamma_1, \ldots, \gamma_n\} \subset \mathcal{P}_\Lambda$. Hence $E_\Lambda\big(\mathrm{e}^{\xi\, S_\Lambda}\big)$ equals

$$\frac{1}{\Xi_\Lambda(\boldsymbol{z})} \sum_{\{\gamma_1,\ldots,\gamma_n\}\subset\mathcal{P}_\Lambda} z_{\gamma_1}\cdots z_{\gamma_n}\, \mathrm{e}^{\xi\,[\alpha(\gamma_1)+\cdots+\alpha(\gamma_n)]} \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

That is,

$$E_\Lambda\big(\mathrm{e}^{\xi\, S_\Lambda}\big) \;=\; \frac{\Xi_\Lambda(\boldsymbol{z}^\xi)}{\Xi_\Lambda(\boldsymbol{z})} \quad \text{with} \quad z_\gamma^\xi = z_\gamma\, \mathrm{e}^{\xi\alpha(\gamma)}$$

Complex ξ are of interest! Also $\xi \to \boldsymbol{\xi}$

Free energy and phase transitions

For (translation-invariant) stat-mech models

$$f(\beta, \boldsymbol{h}) \;=\; \lim_{\Lambda \to \mathbb{L}} \frac{1}{|\Lambda|} \log Z_\Lambda^\sigma$$

exists and is independent of the boundary condition σ

- Spin systems: $-f/\beta$ =free-energy density
- Gas models: f/β = pressure

Derivatives of f yield sums of correlations

Key information: smoothness as function of β and $\boldsymbol{h}$

Loss of analyticity = phase transition (of some sort)

Analiticity radius, zeros and phase transitions

If $\frac{1}{|\Lambda|}\log Z_\Lambda$ has a Λ-independent radius of analyticity at $(\beta, \boldsymbol{h})$:

- No phase transition for $(\beta, \boldsymbol{h})$ within this radius
- Zeros of Z_Λ Λ-uniformly away from $(\beta, \boldsymbol{h})$

For the analyticity of f, one resorts to Vitali's theorem

Let f_n be a sequence of functions, D a domain and S a subset of D containing a accumulation point. If the functions f_n

- *are analytic in D,*
- *are uniformly bounded in D, and*
- *converge pointwisely in S;*

then there exists a function f_∞ such that $f_n \to f_\infty$ uniformly on compacts subsets of D

Alternative lines of attack

Physicist:

Control Ξ through expansion techniques $\longrightarrow$ cluster expansions

- Genesis/reincarnations: Mayer, virial, high-temperature, low-density, . . . expansions
- Not everybody's cup of tea
- Involves algebraic and graph theoretical considerations
- Less natural for purely probabilistic studies (analyticity?)

Probabilists:

Models with exclusions = invariant measures of point processes

- Weaker results (no analyticity!) but wider applicability
- Can use probabilistic techniques (coupling!)
- Leads to (perfect) simulation algorithms

Cluster expansions

The idea is to write the polynomials in $(z_\gamma)_{\gamma\in\mathcal{P}}$

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

as *formal* exponentials of another *formal* series

$$\Xi_\Lambda(\boldsymbol{z}) \stackrel{\mathrm{F}}{=} \exp\Big\{\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \phi^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1} \ldots z_{\gamma_n}\Big\}$$

The series between curly brackets is the *cluster expansion*

Comments

- WATCH OUT!: No consistency requirement, thus

$$\frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \neq \sum_{\{\gamma_1,\ldots,\gamma_n\}\subset\mathcal{P}_\Lambda}$$

- More generally

$$\mathbb{1}_{\{\gamma_j\sim\gamma_k\}} \longrightarrow \varphi(\gamma_j,\gamma_k)$$

for $0 \leq \varphi(\gamma_j,\gamma_k) \leq 1$.

  - This gives rise to level-2 and level-3 set-ups
  - Most of the theory extends to them

Clusters and truncated functions

- $\phi^T(\gamma_1,\ldots,\gamma_n)$: *Ursell* or *truncated* functions (symmetric)
- *Clusters*: Families $\{\gamma_1,\ldots,\gamma_n\}$ s.t. $\phi^T(\gamma_1,\ldots,\gamma_n)\neq 0$
- The formula of ϕ^T will be given later. Highlights:
  - Clusters are *connected* w.r.t. "$\nsim$"
  - $$\phi^T(\gamma)=1 \quad , \quad \phi^T(\gamma,\gamma')=\begin{cases} -1 & \text{if } \gamma\nsim\gamma' \\ 0 & \text{otherwise}\end{cases}$$

Ratios and derivatives

Telescoping, ratios of partitions = product of one-contour ratios

Substracting cluster expansions:

$$\frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}} \overset{\mathrm{F}}{=} \exp\Big\{\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n\\ \exists i:\,\gamma_i=\gamma_0}}\phi^T(\gamma_1,\ldots,\gamma_n)\,z_{\gamma_1}\ldots z_{\gamma_n}\Big\}$$

Slightly more convenient series (proof later):

$$\frac{\partial}{\partial z_{\gamma_0}}\log\Xi_\Lambda \overset{\mathrm{F}}{=} 1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n}\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\,z_{\gamma_1}\ldots z_{\gamma_n}$$

Two strategies to deal with this series: *classical* and *inductive*

Classical cluster-expansion strategy

Find convergence conditions for the series

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;:=\; 1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n}\left|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\right|\,\rho_{\gamma_1}\ldots\rho_{\gamma_n}$$

for $\rho_\gamma > 0$. Then,

Cluster expansions converge *absolutely* for $|z_\gamma| \le \rho_\gamma$ *uniformly in* Λ (complex valued allowed!)

This determines a region of analyticity $\mathcal{R}$ *common for all* Λ

Within this region

$$\frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}} \;\le\; |z_{\gamma_0}|\;\Pi_{\gamma_0}(|\boldsymbol{z}|)$$

Consequences

- Zeros of all Ξ_Λ outside $\mathcal{R}$ (no phase transitions!)
- Within $\mathcal{R}$
  - Explicit series expressions for free energy and correlations
  - Explicit ψ-mixing:

$$\left|\frac{\text{Prob}(\{\gamma_0\,,\,\gamma_x\})}{\text{Prob}(\{\gamma_0\})\,\text{Prob}(\{\gamma_x\})} - 1\right| = \left|\mathrm{e}^{F[d(\gamma_0,\gamma_x)]} - 1\right|$$

    with $F(d) \to 0$ as $d \to \infty$
  - Central limit theorem

Free-energy expansion

Within $\mathcal{R}$

$$\begin{aligned}\log \Xi_\Lambda &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \phi_n^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\ldots z_{\gamma_n} \\ &= \sum_{\gamma\in\mathcal{P}_\Lambda} z_\gamma - \frac{1}{2} \sum_{\substack{(\gamma,\gamma')\in\mathcal{P}_\Lambda^2 \\ \gamma\not\sim\gamma'}} z_\gamma\, z_{\gamma'} + O(|\boldsymbol{z}|^3)\end{aligned}$$

Each term is $O(|\Lambda|)$

Free-energy-density (pressure) expansion

Within $\mathcal{R}$: For the translation-invariant geometrical model

$$f \;=\; \lim_{\Lambda} \frac{1}{|\Lambda|} \log \Xi_\Lambda$$

exists and is analytic on parameters (no phase transitions!)

$$\begin{aligned} f \;&=\; \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\,:\,0\in\cup\gamma_i} \phi_n^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1} \ldots z_{\gamma_n} \\ &=\; \sum_{\gamma\ni 0} z_\gamma - \frac{1}{2} \sum_{\substack{\gamma\nsim\gamma' \\ 0\in\gamma\cup\gamma'}} z_\gamma\, z_{\gamma'} + O(|\boldsymbol{z}|^3) \end{aligned}$$

Correlations

$$\mathrm{Prob}_\Lambda\big(\{\gamma_0\}\big) \;=\; z_{\gamma_0}\,\frac{\Xi_{\Lambda\setminus\{\gamma_0\}^*}}{\Xi_\Lambda} \;=\; z_{\gamma_0}\,\frac{\exp\Big\{\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda\\ \mathcal{C}\sim\gamma_0}} W^T(\mathcal{C})\Big\}}{\exp\Big\{\sum_{\mathcal{C}\subset\mathcal{P}_\Lambda} W^T(\mathcal{C})\Big\}}$$

[$\mathcal{C}\sim\mathcal{C}'$ means $\gamma\sim\gamma'$ for all $\gamma\in\mathcal{C}$, $\gamma'\in\mathcal{C}'$]. Hence

$$\mathrm{Prob}_\Lambda\big(\{\gamma_0\}\big) \;=\; z_{\gamma_0}\,\exp\Big\{\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda\\ \mathcal{C}\nsim\gamma_0}} W^T(\mathcal{C})\Big\}$$

[$\mathcal{C}\nsim\mathcal{C}'$ means $\exists\gamma\in\mathcal{C},\gamma'\in\mathcal{C}'$ with $\gamma\nsim\gamma'$]

$\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda\\ \mathcal{C}\nsim\gamma_0}} W^T(\mathcal{C})$ = cluster expansion *pinned* at γ_0.

Expansion for correlations

Thus

$$\begin{aligned}
&\text{Prob}\big(\{\gamma_0\}\big)\\
&\quad= z_{\gamma_0}\, \exp\Big\{-\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{(\gamma_1,\ldots,\gamma_n)\\ \exists i:\gamma_i\not\sim\gamma_0}}\phi^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\ldots z_{\gamma_n}\Big\}\\
&\quad= z_{\gamma_0}\, \exp\Big\{\sum_{\gamma\not\sim\gamma_0} z_\gamma + O(|\boldsymbol{z}|^2)\Big\}\\
&\quad= z_{\gamma_0}\Big[1+\sum_{\gamma\not\sim\gamma_0} z_\gamma\Big] + O(|\boldsymbol{z}|^3)
\end{aligned}$$

Mixing properties

$$\mathrm{Prob}_\Lambda(\{\gamma_0\} \mid \{\gamma_x\}) = \frac{\mathrm{Prob}_\Lambda(\{\gamma_0\,,\,\gamma_x\})}{\mathrm{Prob}_\Lambda(\{\gamma_x\})}$$

$$= z_{\gamma_0} \frac{\Xi_{\Lambda\setminus\{\gamma_0,\gamma_x\}^*}}{\Xi_{\Lambda\setminus\{\gamma_x\}^*}}$$

Thus,

$$\mathrm{Prob}_\Lambda(\{\gamma_0\} \mid \{\gamma_x\}) = z_{\gamma_0} \frac{\exp\Bigl\{\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda \\ \mathcal{C}\sim\gamma_0,\gamma_x}} W^T(\mathcal{C})\Bigr\}}{\exp\Bigl\{\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda \\ \mathcal{C}\sim\gamma_x}} W^T(\mathcal{C})\Bigr\}}$$

$$= z_{\gamma_0} \exp\Bigl\{-\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda \\ \mathcal{C}\nsim\gamma_0\,,\,\mathcal{C}\sim\gamma_x}} W^T(\mathcal{C})\Bigr\}$$

ψ-mixing

Hence

$$\frac{\mathrm{Prob}_\Lambda(\{\gamma_0\} \mid \{\gamma_x\})}{\mathrm{Prob}_\Lambda(\{\gamma_0\})} = \frac{\exp\Bigl\{-\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda \\ \mathcal{C}\nsim\gamma_0\,,\,\mathcal{C}\sim\gamma_x}} W^T(\mathcal{C})\Bigr\}}{\exp\Bigl\{-\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda \\ \mathcal{C}\nsim\gamma_0}} W^T(\mathcal{C})\Bigr\}}$$

and

$$\begin{aligned}\frac{\mathrm{Prob}(\{\gamma_0\} \mid \{\gamma_x\})}{\mathrm{Prob}(\{\gamma_0\})} &= \exp\Bigl\{\sum_{\substack{\mathcal{C}\subset\mathcal{P}_\Lambda \\ \mathcal{C}\nsim\gamma_0\,,\,\mathcal{C}\nsim\gamma_x}} W^T(\mathcal{C})\Bigr\} \\ &= \mathrm{e}^{F[d(\gamma_0,\gamma_x)]}\end{aligned}$$

with $F(d) \to 0$ as $d \to \infty$. Thus

$$\left|\frac{\mathrm{Prob}(\{\gamma_0\,,\,\gamma_x\})}{\mathrm{Prob}(\{\gamma_0\})\,\mathrm{Prob}(\{\gamma_x\})} - 1\right| = \left|\mathrm{e}^{F[d(\gamma_0,\gamma_x)]} - 1\right|$$

Central Limit Theorem

Lemma (Dobrushin)

Let (S_n) *be a sequence of random variables such that*

(i) $\mathbb{E}(S_n^2) < \infty$

(ii) $\mathrm{Var}(S_n) \geq c\, n$

(iii) $\exists\, R > 0$ *such that*

$$\Big|\log\big|\mathbb{E}(\mathrm{e}^{\xi S_n})\big|\Big| \leq \widetilde{c}\, n \qquad \textit{if } |\xi| < R$$

Then

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \xrightarrow{\text{Law}} \mathcal{N}(0,1)$$

Inductive strategy (Kotecký-Preiss, Dobrushin)

Find conditions on $\mathbf{z}$ defining a region $\mathcal{R}$ such that

$$\Xi_{\Lambda\setminus\{\gamma_0\}^*} \neq 0 \text{ in } \mathcal{R} \quad \Longrightarrow \quad \Xi_\Lambda \neq 0 \text{ in } \mathcal{R}$$

for all Λ, $\gamma_0 \notin \Lambda$

- Expansion neither needed nor obtained (*no-cluster-expansion* method)
- A posteriori: expansion converges in $\mathcal{R}$ $\longrightarrow$ above concl.

Questions raised

- Why the alternative approach leads to better results?
- Can it be interpreted in terms of the classical approach?

Answer: Classical theory revisited