Cluster expansions: Overview and new convergence results

II. Equivalent polymer models;, generalizations, proof of algebraic identities

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The setup

Ingredients

- *Countable* family $\mathcal{P}$ of objects: polymers, animals, ...
- *Incompatibility* constraint: $\gamma \nsim \gamma'$ (with $\gamma \nsim \gamma$)
- *Activities* $\boldsymbol{z} = \{z_\gamma\}_{\gamma\in\mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$.

The basic ("finite-volume") measures

For each *finite* family $\mathcal{P}_\Lambda \subset \mathcal{P}$

$$\mathrm{W}_\Lambda\big(\{\gamma_1,\gamma_2,\ldots,\gamma_n\}\big) \;=\; \frac{1}{\Xi_\Lambda(\boldsymbol{z})}\, z_{\gamma_1} z_{\gamma_2}\cdots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} z_{\gamma_1} z_{\gamma_2}\ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

Graph-theoretical framework

Incompatibility graph $\mathcal{G} = (\mathcal{P}, \mathcal{E})$

- Incompatible = neighboring ($\gamma \nsim \gamma' \;\equiv\; \gamma \leftrightarrow \gamma'$)
- Polymer system = hard-core gas in a complicated lattice
- $\mathcal{N}^*_{\gamma_0} = \{\gamma \in \mathcal{P} : \gamma \nsim \gamma_0\}$; $\mathcal{N}_{\gamma_0} = \mathcal{N}^*_{\gamma_0} \setminus \{\gamma_0\}$
- *Independent vertices* = non-neighboring vertices
- *Independent sets* = sets formed by independent vertices

Thus,

$$\Xi_\Lambda(\boldsymbol{z}) \;=\; \sum_{\substack{\Gamma \subset \mathcal{P}_\Lambda \\ \text{independent}}} \boldsymbol{z}^\Gamma \qquad \text{with} \quad \boldsymbol{z}^\Gamma = \prod_{\gamma \in \Gamma} z_\gamma$$

Ratios of partition functions

- Correlations:

$$\mathrm{Prob}_\Lambda\big(\{\gamma_1,\ldots,\gamma_k \text{ are present}\}\big) \;=\; z_{\gamma_1}\cdots z_{\gamma_k}\,\frac{\Xi_{\Lambda\setminus\{\gamma_1,\ldots,\gamma_k\}^*}}{\Xi_\Lambda}$$

- Characteristic functions: If $S_\Lambda(\gamma_1,\ldots,\gamma_n) = \sum_{i=1}^n \alpha(\gamma_i)$

$$E_\Lambda\big(\mathrm{e}^{\xi\, S_\Lambda}\big) \;=\; \frac{\Xi_\Lambda(\boldsymbol{z}^\xi)}{\Xi_\Lambda(\boldsymbol{z})} \quad \text{with} \quad z_\gamma^\xi = z_\gamma\,\mathrm{e}^{\xi\alpha(\gamma)}$$

- Zeros of partition functions related to smoothness of

$$f(\beta,\boldsymbol{h}) \;=\; \lim_{\Lambda\to\mathbb{L}} \frac{1}{|\Lambda|}\,\log Z_\Lambda^\sigma$$

Previous example: Single-call loss networks

- $\mathcal{P}$ = finite connected families of links of $\mathbb{Z}^d$ —the *calls*
- z_γ = Poissonian rate for the call γ
- Compatibility = use of disjoint links (no intersection)
- Basic measures are invariant for the finite-region process
- Thermodynamic limit: infinite-volume process

Previous example: Ising model at low T

Using the *contour representation*:

- Polymers = contours (connected closed surfaces)
- Compatibility = no intersection
- $z_\gamma = \exp\{-2\beta J \ |\gamma|\}$

Then

$$W_\Lambda(\omega \mid +) \;=\; \frac{1}{\Xi_\Lambda} \prod_{\gamma \in \Gamma(\omega)} z_\gamma$$

with

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n) \in \mathcal{C}_\Lambda^n} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

Previous example: LTE for Ising ferromagnets

Write

$$H_\Lambda(\omega) \;=\; -\sum_{B\in\mathcal{B}_\Lambda} J_B\big(\omega^B - 1\big) - \sum_{B\in\mathcal{B}_\Lambda} J_B$$

- *Contour* = connected component of (excited) bonds
- $z_\gamma \;=\; \exp\big\{-2\beta\sum_{B\in\gamma} J_B\big\}$
- $\gamma \sim \gamma'$ iff $\underline{\gamma}\cap\underline{\gamma'} = \emptyset$ (disjoint bases); $\underline{\gamma} = \cup\{B : B\in\gamma\}$

Then $Z_\Lambda = |\mathcal{S}_\Lambda|\;\Xi_\Lambda^{\rm LT}$ with

$$\mathcal{S}_\Lambda \;=\; \big\{\chi : \chi^B = 1 \text{ for all } B\in\mathcal{B}_\Lambda\big\}$$

(symmetry group) and

$$\Xi_\Lambda^{\rm LT}(\boldsymbol{z}) = 1 + \sum_{n\ge 1}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{C}_\Lambda^n} z_{\gamma_1}z_{\gamma_2}\ldots z_{\gamma_n}\prod_{j<k}\mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

Geometrical polymer models

Original polymer models of Gruber and Kunz:

- $\mathcal{P}$ = family of finite subsets of some set $\mathbb{V}$
- $\gamma \sim \gamma' \Longleftrightarrow \gamma \cap \gamma' = \emptyset$

Usually

- $\mathbb{V}$ = vertex set of a graph (lattice, dual lattice)
- Polymers defined by connectivity properties
- Compatibility determined by graph distances

Warning: Do not confuse with the incompatibility graph

A little more general: decorated geometrical polymers

$$\gamma = (\underline{\gamma}, D_\gamma) \quad , \quad \underline{\gamma} = \text{“base”} \subset\subset \mathbb{V} \, , \; D_\gamma = \text{ “decoration”}$$

Cluster expansions

Write the polynomials (in $(z_\gamma)_{\gamma\in\mathcal{P}}$)

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

as *formal* exponentials of a *formal* series

$$\Xi_\Lambda(\boldsymbol{z}) \overset{\mathrm{F}}{=} \exp\Big\{ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \phi^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1} \ldots z_{\gamma_n} \Big\}$$

- The series between curly brackets is the *cluster expansion*
- $\phi^T(\gamma_1,\ldots,\gamma_n)$: *Ursell* or *truncated* functions (symmetric)
- *Clusters*: Families $\{\gamma_1,\ldots,\gamma_n\}$ s.t. $\phi^T(\gamma_1,\ldots,\gamma_n) \neq 0$
- Clusters are *connected* w.r.t. "$\nsim$"

Classical cluster-expansion strategy

Find a Λ-*independent* polydisc where cluster expansions converge *absolutely*

That is, find $\rho_\gamma > 0$ independent of Λ such that cluster expansions converge absolutely in the region

$$\mathcal{R} \;=\; \Big\{ \boldsymbol{z} : |z_\gamma| \le \rho_\gamma \,,\, \gamma \in \mathcal{P} \Big\}$$

To this, find $\boldsymbol{\rho} > 0$ such that

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;:=\; 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \Big|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\Big| \,\rho_{\gamma_1}\ldots\rho_{\gamma_n}$$

converges. Within this region

- No Ξ_Λ has a zero
- Explicit series expressions for free energy and correlations
- Explicit ψ-mixing
- Central limit theorem

Part III

Associated polymer models. Generalizations

Associated polymer model = same partition ratios

More precisely,

$$Z_\Lambda^{\text{model}}(\text{param.}) \;=\; \text{const}_\Lambda \, \Xi_\Lambda^{\text{polymer}}(\boldsymbol{z})$$

$(\text{const}_\Lambda \sim a^{|\Lambda|})$.

Let us review several manifestations of this association

Outline

Useful observation: Distributivity property

If S finite set and $(\varphi_a)_{a\in S}$, $(\psi_a)_{a\in S}$ complex-valued:

$$\prod_{a\in S}[\psi_a + \varphi_a] = \sum_{A\subset S} \prod_{a\in A} \varphi_a \prod_{a\in S\setminus A} \psi_a$$

$[\prod_{\emptyset} \equiv 1]$

Models at high temperature

$$\exp\Big\{-\beta \sum_{A\in\mathcal{B}_\Lambda} \phi_A(\omega)\Big\} = \prod_{A\in\mathcal{B}_\Lambda}\Big[1+\big(e^{-\beta\,\phi_A(\omega)}-1\big)\Big]$$

$$= \sum_{\boldsymbol{B}\subset\mathcal{B}_\Lambda}\prod_{A\in\boldsymbol{B}}\big(e^{-\beta\,\phi_A(\omega)}-1\big)$$

Separating $\boldsymbol{B}$ into connected (w.r.t. overlapping) components,

$$Z_\Lambda = \sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{(\boldsymbol{B}_1,\ldots,\boldsymbol{B}_n)\subset\boldsymbol{B}_\Lambda^n\\ \boldsymbol{B}_i\ \text{conn.}}}\prod_{i=1}^{n}\Big[\int_{\underline{\boldsymbol{B}_i}}\prod_{A\in\boldsymbol{B}_i}\big(e^{-\beta\,\phi_A(\omega)}-1\big)\bigotimes_{x\in\cup\underline{\boldsymbol{B}_i}}\mu_E(d\omega_x)\Big]$$

$$\times\prod_{i<j}\mathbb{1}_{\{\underline{\boldsymbol{B}_i}\cap\underline{\boldsymbol{B}_j}=\emptyset\}}$$

$[\underline{\boldsymbol{B}} =$ support of $\boldsymbol{B} = \cup\{B : B\in\boldsymbol{B}\}]$

High-temperature expansion

Hence

$$Z_\Lambda \;=\; \Xi_\Lambda^{\mathrm{HT}}$$

for the polymer system with

- $\mathcal{P} = \{$connected finite subsets of bonds$\}$
- $\boldsymbol{B} \sim \boldsymbol{B}'$ iff $\underline{\boldsymbol{B}} \cap \underline{\boldsymbol{B}}' = \emptyset$
- $$z_{\boldsymbol{B}} = \int_{\underline{\boldsymbol{B}}} \prod_{A \in \boldsymbol{B}} (\mathrm{e}^{-\beta\,\phi_A(\omega)} - 1) \bigotimes_{x \in \underline{\boldsymbol{B}}} \mu_E(d\omega_x)$$

  (small at small β, i.e. large temperature)

Corresponding cluster expansion = *high-temperature expansion*

HTE for Ising ferromagnets

Obtained by exploiting in

$$Z_\Lambda = \sum_{\omega_\Lambda} \prod_{B\in\mathcal{B}_\Lambda} \mathrm{e}^{\beta J_B \omega^B}$$

the observation

$$\mathrm{e}^{\beta J_B \omega^B} = \cosh(\beta J_B) + \omega^B \sinh(\beta J_B)$$

to get

$$\begin{aligned} Z_\Lambda &= \Big[\prod_{B\in\mathcal{B}_\Lambda} \cosh(\beta J_B)\Big] \sum_{\omega_\Lambda} \prod_{B\in\mathcal{B}_\Lambda} \Big[1 + \omega^B \tanh(\beta J_B)\Big] \\ &= \Big[\prod_{B\in\mathcal{B}_\Lambda} \cosh(\beta J_B)\Big] \sum_{\boldsymbol{B}\subset\mathcal{B}_\Lambda} \sum_{\omega_\Lambda} \prod_{B\in\boldsymbol{B}} \omega^B \tanh(\beta J_B) \end{aligned}$$

Group of cycles

But

$$\prod_{B\in\boldsymbol{B}} \omega^B \;=\; \omega^{\sum_{B\in\boldsymbol{B}}}$$

with $\sum$ =symmetric difference, and

$$\sum_{\omega_\Lambda} \omega^B \;=\; \begin{cases} 2^{|\Lambda|} & \text{if } B = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$Z_\Lambda \;=\; 2^{|\Lambda|}\Big[\prod_{B\in\mathcal{B}_\Lambda} \cosh(\beta J_B)\Big] \sum_{\boldsymbol{B}\subset\mathcal{K}_\Lambda} \prod_{B\in\boldsymbol{B}} \tanh(\beta J_B)$$

with

$$\mathcal{K}_\Lambda \;=\; \Big\{\boldsymbol{B}\in\mathcal{B}_\Lambda : \sum_{B\in\boldsymbol{B}} B = \emptyset\Big\}$$

Ferromagnetic HT polymer model

The maximally *connected* elements of $\mathcal{K}_\Lambda$ are the *cycles*

($\mathcal{K}_\Lambda$ is a group for "$\sum$", generated by the cycles)

Factorizing the contribution of cycles,

$$Z_\Lambda \;=\; 2^{|\Lambda|}\Big[\prod_{B\in\mathcal{B}_\Lambda}\cosh(\beta J_B)\Big]\,\Xi_\Lambda^{\mathrm{IHT}}$$

for the polymer system with

- Polymers $\mathcal{P} = \{$ cycles$\}$
- Consistency: $\boldsymbol{B} \sim \boldsymbol{B}'$ iff $\underline{\boldsymbol{B}} \cap \underline{\boldsymbol{B}'} = \emptyset$
- Fugacities (small at small β)

$$z_{\boldsymbol{B}} = \prod_{B\in\boldsymbol{B}} \tanh(\beta J_B)$$

LTE vs HTE for Ising ferromagnets

$$Z_\Lambda = |\mathcal{S}_\Lambda| \left[\prod_{B \in \mathcal{B}_\Lambda} e^{2J_B} \right] \Xi_\Lambda^{\mathrm{LT}}(\boldsymbol{z}^{\mathrm{LT}})$$

$$Z_\Lambda = 2^{|\Lambda|} \left[\prod_{B \in \mathcal{B}_\Lambda} \cosh(\beta J_B) \right] \Xi_\Lambda^{\mathrm{HT}}(\boldsymbol{z}^{\mathrm{HT}})$$

($\mathcal{S}_\Lambda$ = symmetry group = $\{\chi : \chi^B = 1 \text{ for all } B \in \mathcal{B}_\Lambda\}$)

$$\mathcal{P}_\Lambda^{\mathrm{LT}} = \mathcal{C}_\Lambda = \{\text{contours}\} \quad , \quad \mathcal{P}_\Lambda^{\mathrm{HT}} = \mathcal{K}_\Lambda = \{\text{cycles}\}$$

(contour = connected set of excited *bonds*, cycle = set of *bonds* covering each site an even number of times)

$$z_{\boldsymbol{B}}^{\mathrm{LT}} = \exp\left\{ -2\beta \sum_{B \in \boldsymbol{B}} J_B \right\}$$

$$z_{\boldsymbol{B}}^{\mathrm{HT}} = \prod_{B \in \boldsymbol{B}} \tanh(\beta J_B)$$

HT–LT duality

Let us absorb β into the couplings J_B

$\big(\Lambda^*, \mathcal{B}^*_\Lambda, (J^*_B)_{B\in\mathcal{B}^*_\Lambda}\big)$ is the *HT–LT dual* of $\big(\Lambda, \mathcal{B}_\Lambda, (J_B)_{B\in\mathcal{B}_\Lambda}\big)$ if there exists a surjective map $D : \mathcal{B}_\Lambda \to \mathcal{B}^*_\Lambda$ such that

(i) The map

$$\begin{aligned} \overline{D} : \mathcal{P}(\mathcal{B}_\Lambda) &\longrightarrow \mathcal{P}(\mathcal{B}^*_\Lambda) \\ \overline{D}(\boldsymbol{B}) &= \cup_{B\in\boldsymbol{B}} D(B) \end{aligned}$$

induces a surjection (bijection) $\mathcal{K}_\Lambda \to \mathcal{C}^*_\Lambda$

(ii) For each $B^* \in \mathcal{B}^*_\Lambda$

$$\mathrm{e}^{-2J^*_{B^*}} = \prod_{B\in D^{-1}(B^*)} \tanh(J_B)$$

Dual systems

For HT–LT duals

$$Z_\Lambda \;=\; 2^{|\Lambda|}\;|\mathcal{S}|^{-1}\Big[\prod_{B\in\boldsymbol{B}}\cosh(J_B)\Big]\Big[\prod_{B^*\in\boldsymbol{B}^*}\mathrm{e}^{-2J^*_{B^*}}\Big]\,Z^*_{\Lambda^*}$$

Hence:

$$\text{convergent C.E. for } Z^*_{\Lambda^*} \Longleftrightarrow \text{convergent C.E. for } Z_\Lambda$$

That is,

$$\text{analyticity of } f^* \Longleftrightarrow \text{analyticity of } f$$

Construction of HT–LT duals

- Consider a family $\{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_k\}$ of generators of $\mathcal{K}_\Lambda$
- Associate to each $\boldsymbol{B}_i$ a site $x_i^* \in \Lambda^*$
- Define

$$D(B) \;=\; \{x_i^* : \boldsymbol{B}_i \ni B\}$$

In particular

- Regular 2-d Ising is self-dual
- Ising with four body has as dual the usual Ising

Comments

- *Strong duality*: $\mathcal{K}_\Lambda = \mathcal{C}_\Lambda^*$
- Similarly there are LT–HT, HT–HT and LT–LT duals

Potts model

Let

- $(\mathbb{L}, \mathcal{B})$ be a graph any (eg. $\mathbb{L} = \mathbb{Z}^d$, $\mathcal{B}$ =n.n. pairs),
- $E = \{1, \ldots, q\}$, $\mathcal{F}$ =discrete, μ_E =counting
- $$\phi_B(\omega) \;=\; \begin{cases} -J_{x\,y}\left(\delta_{\omega_x \omega_y} - 1\right) & \text{if } \{x,y\} \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$
  - $\phi_{\{x,y\}} = J$ if $\omega_x \neq \omega_y$, 0 otherwise
  - If $q = 2$, Potts=Ising

Finally,

$$Z_\Lambda^{\text{Potts}}(\beta, q) \;=\; \sum_{\omega_\Lambda} \prod_{\{x,y\} \in \mathcal{B}_\Lambda} \mathrm{e}^{\beta J_{x\,y}(\delta_{\omega_x \omega_y} - 1)}$$

The FK trick

Crucial observation:

$$\begin{aligned} e^{\beta J_{x\,y}(\delta_{\omega_x\omega_y}-1)} &= \delta_{\omega_x\omega_y} + e^{-\beta J_{x\,y}}(1-\delta_{\omega_x\omega_y}) \\ &= (1-p_{x\,y}) + p_{x\,y}\,\delta_{\omega_x\omega_y} \end{aligned}$$

with $p_{x\,y} = 1 - e^{-\beta J_{x\,y}}$. Hence, for $\Lambda \subset\subset \mathbb{L}$,

$$\begin{aligned} Z_\Lambda^{\text{Potts}}(\beta,q) &= \sum_{\omega_\Lambda} \prod_{\{x,y\}\in\mathcal{B}_\Lambda} \Big[(1-p_{x\,y}) + p_{x\,y}\,\delta_{\omega_x\omega_y}\Big] \\ &= \sum_{\omega_\Lambda} \sum_{\boldsymbol{B}\subset\mathcal{B}_\Lambda} \prod_{\{x,y\}\in\boldsymbol{B}} \delta_{\omega_x\omega_y} \prod_{\{x,y\}\in\boldsymbol{B}} p_{x\,y} \prod_{\{x,y\}\notin\boldsymbol{B}} (1-p_{x\,y}) \end{aligned}$$

($\mathcal{B}_\Lambda$ = links (bonds) with vertices in Λ)

The FK expansion

As

$$\sum_{\omega_\Lambda} \prod_{\{x,y\}\in \boldsymbol{B}} \delta_{\omega_x \omega_y} \;=\; q^{C(\boldsymbol{B})}$$

with $C(\boldsymbol{B}) = \#$ connected components of $\boldsymbol{B}$,

$$Z_\Lambda^{\text{Potts}}(\beta, q) \;=\; \sum_{\boldsymbol{B}\subset \mathcal{B}_\Lambda} q^{C(\boldsymbol{B})} \prod_{\{x,y\}\in \boldsymbol{B}} p_{x\,y} \prod_{\{x,y\}\notin \boldsymbol{B}} (1 - p_{x\,y})$$

- $q = 1$: regular (independent) bond percolation in $\mathbb{Z}^d$
- $q > 1$: dependent percolation due to $q^{C(\boldsymbol{B})}$

FK model

$$Z_\Lambda^{\text{Potts}}(\beta, q) = \Big[\prod_{\{x,y\}\in\mathcal{B}_\Lambda} (1 - p_{x\,y})\Big] \sum_{\boldsymbol{B}\subset\mathcal{B}_\Lambda} q^{C(\boldsymbol{B})} \prod_{\{x,y\}\in\boldsymbol{B}} \frac{p_{x\,y}}{1 - p_{x\,y}}$$

$$= \Big[\prod_{\{x,y\}\in\mathcal{B}_\Lambda} (1 - p_{x\,y})\Big] Z_\Lambda^{\text{FK}}(q, \boldsymbol{v})$$

with

$$Z_\Lambda^{\text{FK}}(q, \boldsymbol{v}) = \sum_{\boldsymbol{B}\subset\mathcal{B}_\Lambda} q^{C(\boldsymbol{B})} \prod_{\{x,y\}\in\boldsymbol{B}} v_{x\,y}$$

and

$$v_{x\,y} = \frac{p_{x\,y}}{1 - p_{x\,y}} = \mathrm{e}^{\beta\,J_{x\,y}} - 1$$

FK polymer model

(Also called *random-cluster model*)

Reorder the sum:

- Each $\boldsymbol{B}$ defines a graph $G = (V_{\boldsymbol{B}}, \boldsymbol{B})$
- Let $G_i = (V_i, \boldsymbol{B}_i)$, $i = 1, \ldots, k$ connected components
  - The vertex sets are disjoints: $V_i \cap V_j = \emptyset$ if $i \neq j$
  - The sets of bonds $\boldsymbol{B}_i$ are such that each G_i is connected

Furthermore

$$\begin{aligned} C(\boldsymbol{B}) &= k + \#\text{ isolated points} \\ &= k + |\Lambda| - \sum |V_i| \\ &= |\Lambda| - \sum (|V_i| - 1) \end{aligned}$$

High-q expansion

Then

$$\frac{Z_\Lambda^{\mathrm{FK}}(q,\boldsymbol{v})}{q^{|\Lambda|}} = \sum_{k\geq 0}\frac{1}{k!}\sum_{\substack{(V_1,\ldots,V_k)\subset\Lambda^k\\ \text{disjoints}}}\prod_{i=1}^{k}\left[q^{-(|V_i|-1)}\sum_{\substack{\boldsymbol{B}_i\subset\mathcal{B}_{V_i}\\ (V_i,\boldsymbol{B}_i)\text{ conn.}}}\prod_{\{x,y\}\in\boldsymbol{B}_i} v_{x\,y}\right]$$

$$= \Xi_\Lambda^{\mathrm{FK}}(\boldsymbol{z})$$

FK geometrical polymer system: $\mathcal{P} = \{V \subset\subset \mathbb{L}\}$,

$$z_V = q^{-(|V|-1)} \sum_{\substack{\boldsymbol{B}\subset\mathcal{B}_V\\ (V,\boldsymbol{B})\text{ connected}}}\prod_{\{x,y\}\in\boldsymbol{B}} v_{x\,y}$$

decreases as $q\to\infty$ (or as $\beta\to 0$)

Corresponding cluster expansion = *high-q (high-T) expansion*

Chromatic polynomials

Given a graph $G = \big(V(G), E(G)\big)$:

$$P_G(q) \;=\; \#\text{ ways of properly coloring } G \text{ with } q \text{ colors}$$

"properly" = adjacents vertices have different colors

If $\omega : V(G) \to \{1, \ldots, q\}$ denote colorings

$$P_G(q) \;=\; \sum_{\omega} \prod_{\{x,y\} \in E(G)} \left[1 - \delta_{\omega_x\,\omega_y}\right]$$

Introduced by Birkhoff (1912) to determine

$$\chi_G \;=\; \min\big\{q : P_G(q) > 0\big\}$$

chromatic number = minimal q for a proper coloring

Tutte polynomial

Slight generalization: $(-1) \to v_{x\,y}$

$$P_G(q, \boldsymbol{v}) = \sum_{\omega} \prod_{\{x,y\} \in E(G)} \Big[1 + v_{x\,y}\, \delta_{\omega_x\, \omega_y} \Big]$$
$$= \sum_{\boldsymbol{B} \subset E(G)} q^{C(\boldsymbol{B})} \prod_{\{x,y\} \in \boldsymbol{B}} v_{x\,y}$$

This is a multivariate version of the Tutte polynomial

For us

$$P_G(q, \boldsymbol{v}) = Z_\Lambda^{\mathrm{FK}}(q, \boldsymbol{v}) = q^{|\Lambda|}\, \Xi_\Lambda^{\mathrm{FK}}(\boldsymbol{z})$$

This identity proves that $P_G(q, \boldsymbol{v})$ is a polynomial in q

Standard Tutte polynomial

Its original definition is

$$\begin{aligned} T_G(\lambda, \mu) &= \sum_{\boldsymbol{B} \subset \mathcal{E}} (\lambda - 1)^{C(\boldsymbol{B}) - C(E)} (\mu - 1)^{|\boldsymbol{B}| + C(\boldsymbol{B}) - |V|} \\ &= (\lambda - 1)^{-C(E)} (\mu - 1)^{-|V|} P_G\big((\lambda - 1)(\mu - 1), \mu - 1\big) \end{aligned}$$

Important properties:

- It is related with the
  - flow polynomial
  - dichromatic polynomial
  - Whitney rank function
- Different limits yield generating polynomials:

Multivariate generating polynomials

$$\lim_{q\to 0} q^{-C(E)} P_G(q, \boldsymbol{v}) = \sum_{\substack{\boldsymbol{B}\subset E \\ \text{conn. spann.}}} \boldsymbol{v}^{\boldsymbol{B}}$$

$$\lim_{q\to 0} q^{-|V|} P_G(q, q\boldsymbol{v}) = \sum_{\substack{\boldsymbol{B}\subset E \\ \text{spann. forest}}} \boldsymbol{v}^{\boldsymbol{B}}$$

$$\lim_{t\to 0} t^{C(E)-|V|} \left[\lim_{q\to 0} q^{-C(E)} P_G(q, t\boldsymbol{v})\right] = \sum_{\substack{\boldsymbol{B}\subset E \\ \text{max. spann. forest}}} \boldsymbol{v}^{\boldsymbol{B}}$$

RHS are multivariate generating polynomials of

- spanning graphs
- spanning forests
- maximal spanning forests (= spanning trees if G connected)

Chromatic numbers and cluster expansions

If $J_{x\,y} < 0$ (antiferromagnetic Potts model)

$$v_{x\,y} = \mathrm{e}^{\beta J_{x\,y}} - 1 \xrightarrow[\beta\to\infty]{} -1$$

Hence

$$P_G(q) \;=\; Z_\Lambda^{\mathrm{FK}}(q,-1) \;=\; q^{|\Lambda|}\, \Xi_\Lambda^{\mathrm{FK}}(\boldsymbol{z}^-)$$

with

$$z_V^- \;=\; q^{-(|V|-1)} \sum_{\substack{\mathbf{B}\subset\mathcal{B}_{\mathbf{V}} \\ (\mathbf{V},\boldsymbol{B})\,\mathrm{conn.}}} (-1)^{|\boldsymbol{B}|}$$

Region free the zeros of $P_G(q) \to$ bound on χ_G

Inhomogeneous Markov chains

Let $(X_n)_{n\geq 0}$ be a Markov chain, $X_n : \Omega \to E$, characterized by

$$\begin{aligned} p_n(x_{n-1}, x_n) &= \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}) \\ p_0(x) &= \mathbb{P}(X_0 = x) \end{aligned}$$

Denote

$$\begin{aligned} p_{[0,n]}(x_0^n) &= p_0(x_0)\, p_1(x_0, x_1) \cdots p_n(x_{n-1}, x_n) \\ p_{[a+1,b]}(x_a^b) &= p_{a+1}(x_a, x_{a+1}) \cdots p_b(x_{b-1}, x_b) \qquad (a > 0) \end{aligned}$$

Consider $\alpha : E \to \mathbb{R}$,

$$S_n(x_0^n) = \sum_{i=0}^{n} \alpha(x_i)$$

and the characteristic function

$$\phi_n(\xi) = \sum_{x_0^n} p_{[0,n]}(x_0^n)\, \mathrm{e}^{\xi S_n(x_0^n)}$$

Polymer representation (Dobrushin)

$$\begin{aligned} e^{\xi S_n(x_0^n)} &= \prod_{i=0}^{n} \Big[1 + \big(e^{\xi\,\alpha(x_i)} - 1 \big) \Big] \\ &= \sum_k \prod_{\substack{[a_1,b_1],\cdots,[a_k,b_k] \\ 0 \le a_i \le b_i \le n\ ,\ b_i < a_{i+1} - 1}} \prod_{\ell=a_i}^{b_i} \Big(e^{\xi\,\alpha(x_\ell)} - 1 \Big) \end{aligned}$$

Hence

$$\phi_n(\xi) = \sum_{x_{\underline{a},\underline{b}}} \sum_{[a_1,b_1],\cdots,[a_k,b_k]} p_{a_1} \chi_{[a_1,b_1]} p_{[b_1,a_2]} \cdots \chi_{[a_k,b_k]} p_{[b_k,n]}$$

Polymer representation (Dobrushin)

where

$$\begin{aligned} p_a(x_a) &= \mathbb{P}(X_a = x_a) \\ p_{[a,b]}(x_a, x_b) &= \sum_{x_{a+1}^{b-1}} p_{[a+1,b]}(x_a^b) \\ \chi_{[a,b]}(x_a, x_b) &= \sum_{x_{a+1}^{b-1}} p_{[a+1,b]}(x_a^b) \prod_{i=a}^{b} \Big(\mathrm{e}^{\xi\, \alpha(x_i)} - 1 \Big) \end{aligned}$$

$\chi_{[a,b]}$ is small if $|\xi|$ is small

Must include relaxation properties of the chain:

$$p_{[b,a]} = \left[p_{[b,a]} - p_a \right] + p_a$$

Polymer representation (Dobrushin)

Result:

$$\phi_n(z) \;=\; \Xi_{[0,n]}$$

with

$$\mathcal{P} \;=\; \Big\{(\underline{a},\underline{b}) = (a_1,b_1,a_2,b_2,\ldots,a_k,b_k) : 0 \le a_i \le b_i \le n\;,\; b_i < a_{i+1}\Big\}$$

and

$$z_{(\underline{a},\underline{b})} \;=\; \sum_{x_{\underline{a}},x_{\underline{b}}} p_{a_1}\,\chi_{[a_1,b_1]}\,\big[p_{[b_1,a_2]} - p_{a_2}\big]\,\chi_{[a_2,b_2]}\,\cdots\,\big[p_{[b_{k-1},a_k]} - p_{a_k}\big]\,\chi_{[a_k,b_k]}$$

Small if $\big|p_{[b,a]} - p_a\big|$ and $|\xi|$ small. Relaxation $\rightarrow$ CLT

Generalization I: Continuous polymer systems

More generally,

$$\frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \longrightarrow \frac{1}{n!}\int_{\mathcal{P}_\Lambda^n} d\gamma_1\cdots d\gamma_n$$

where $d\gamma_1\cdots d\gamma_n$ is an appropriate product measure

Also, for book-keeping purposes: $z_\gamma = z\,\xi_\gamma$

That is, we consider measures on $\sum_n \mathcal{P}_\Lambda^n$ with projections on $\mathcal{P}_\Lambda^n$

$$\frac{1}{\Xi_\Lambda}\,\frac{z^n}{n!}\,\xi_{\gamma_1}\xi_{\gamma_2}\cdots\xi_{\gamma_n}\prod_{j<k}\mathbb{1}_{\{\gamma_j\sim\gamma_k\}}\,d\gamma_1\cdots d\gamma_n$$

where

$$\Xi_\Lambda(z,\boldsymbol{\xi}) = 1+\sum_{n\geq 1}\frac{z^n}{n!}\int_{\mathcal{P}_\Lambda^n}\xi_{\gamma_1}\ldots\xi_{\gamma_n}\prod_{j<k}\mathbb{1}_{\{\gamma_j\sim\gamma_k\}}\,d\gamma_1\cdots d\gamma_n$$

Correlations and cluster expansions

The correlation functions are probability densities —with respect to $d\gamma_1 \cdots d\gamma_n$— of finding polymers $\gamma_1, \ldots, \gamma_n$:

$$\rho_\Lambda(\gamma_1, \ldots, \gamma_n) \;=\; z_{\gamma_1} \ldots z_{\gamma_n} \, \frac{\Xi_{\mathcal{P}_\Lambda \setminus \{\gamma_1, \ldots, \gamma_k\}^*}}{\Xi_\Lambda}$$

The cluster expansion is the formal series such that

$$\Xi_\Lambda \;\stackrel{\mathrm{F}}{=}\; \exp\Bigl\{\sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} \phi^T(\gamma_1, \ldots, \gamma_n)\, \xi_{\gamma_1} \ldots \xi_{\gamma_n}\, d\gamma_1 \cdots d\gamma_n\Bigr\}$$

Interest focuses in appropriate limits $\Lambda \to \infty$, when $\mathcal{P} \to \mathcal{P}_\Lambda$

Example: Classical continuous gas

Basic setting

- Particles moving in a continuous space $\mathbb{S}$ (e.g. $\mathbb{S} = \mathbb{R}^d$)
- Initially particles in a box $\Lambda \subset\subset \mathbb{S}$, eventually $\Lambda \to \mathbb{S}$
- Particles are distinguishable, but interest focuses on which points are occupied and not by whom

Hence:

- Configuration: momenta and positions of particles in a box
- There is a $1/n!$ factor averaging permutations among sites

Ingredients of a continuous systems

- *Energy* of n particules of momenta p_i and positions x_i:

$$H(p_1, \ldots, p_n, x_1, \ldots, x_n) \;=\; \sum_{i=1}^{n} \frac{p_i^2}{2m} + U(x_1, \ldots, x_n)$$

  where U is the *configurational Hamiltonian*

$$U(x_1, \ldots, x_n) \;=\; \sum_{A \subset \{1,\ldots,n\}} \phi_{|A|}\big((x_i)_{i \in A}\big)$$

- *Gibbs chemical potential* μ (acts as a “field”)

Grand canonical ensemble

Measures on $\sum_n \big[(\mathbb{R}^d)^n \times \Lambda^n\big]$ (with $\Lambda \subset\subset \mathbb{S}$), s.t. projected on $(\mathbb{R}^d)^n \times \Lambda^n$:

$$\frac{1}{\widetilde{Z}_\Lambda} \frac{\mathrm{e}^{\beta\mu n}}{n!} \prod_{i=1}^{n} \Big[\exp\Big(-\beta \frac{p_i^2}{2m}\Big) dp_i\Big] \exp\Big[-\beta\, U(x_1, \ldots, x_n)\Big] dx_1 \cdots dx_n$$

with

$$\begin{aligned} \widetilde{Z}_\Lambda \;=\; & \sum_{n\geq 0} \frac{\mathrm{e}^{\beta\mu n}}{n!} \prod_{i=1}^{n} \Big[\int_{\mathbb{R}^d} \exp\Big(-\beta \frac{p_i^2}{2m}\Big) dp_i\Big] \\ & \times \int_{\Lambda^n} \exp\Big[-\beta\, U(x_1, \ldots, x_n)\Big] dx_1 \cdots dx_n \end{aligned}$$

Configurational ensemble

If no questions on momenta,

$$\int_{\mathbb{R}^d} \exp\Big(-\beta \frac{p_i^2}{2m}\Big) dp_i \;=\; \Big(\frac{2\pi m}{\beta}\Big)^{d/2}$$

and ensemble reduces to a measure on $\sum_n \Lambda^n$ with projections

$$\frac{1}{Z_\Lambda} \frac{z^n}{n!} \exp\Big[-\beta\, U(x_1,\ldots,x_n)\Big] dx_1 \cdots dx_n$$

with

$$Z_\Lambda \;=\; \sum_{n\geq 0} \frac{z^n}{n!} \int_{\Lambda^n} \exp\Big[-\beta\, U(x_1,\ldots,x_n)\Big] dx_1 \cdots dx_n$$

and

$$z \;=\; \mathrm{e}^{\beta\mu} \Big(\frac{2\pi m}{\beta}\Big)^{d/2}$$

Correlation functions

Let us abbreviate: $x_k^\ell := x_k, \ldots, x_\ell$

The *m-point correlation function* is the probability density

$$\rho(x_1^n) \;=\; \frac{1}{Z_\Lambda} \sum_{n \geq 0} \frac{z^{m+n}}{n!} \int_{\Lambda^n} \exp\Big[-\beta\, U(x_1^m, y_{m+1}^{n+m})\Big]\, dy_{m+1}^{m+n}$$

Gas of hard spheres

Points = centers of spheres of diameter R:

$$\phi_n(x_1, \ldots, x_n) = \begin{cases} \infty & \text{if } n = 2 \text{ and } |x_1 - x_2| \leq R \\ 0 & \text{otherwise} \end{cases}$$

This gives a continuous polymer system with

- Polymers = centers of spheres in Λ:

$$\mathcal{P} = \mathcal{P}_\Lambda = \{x \in \Lambda : \mathrm{dist}(x, \mathbb{S} \setminus \Lambda) > R/2\}$$

- Compatibility = non-intersection of spheres

$$x \nsim y \iff |x - y| \leq R$$

Generalization II: polymers with soft interactions

Archetypical example: Gas with two-body interaction:

$$U(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} \phi(x_i, x_j)$$

Hence,

$$Z_\Lambda = \sum_{n \ge 0} \frac{z^n}{n!} \sum_{(x_1,\ldots,x_n) \in \Lambda^n} \prod_{1 \le i \le j \le n} \mathrm{e}^{-\beta \phi(x_i, x_j)}$$

or

$$\Xi = 1 + \sum_{n \ge 1} \frac{z^n}{n!} \int_{\Lambda^n} \prod_{1 \le i \le j \le n} \mathrm{e}^{-\beta \phi(x_i, x_j)} \, dx_1 \cdots dx_n$$

Interacting polymer systems

Multivariate ($z \to \boldsymbol{z}$) abstract version. Ingredients:

- A measure space $(\mathcal{P}, d\gamma)$
- Subsets $\mathcal{P}_\Lambda \subset \mathcal{P}$ of finite measure s.t. $\mathcal{P}_\Lambda \to \mathcal{P}$
- An activity parameter z
- One-contour factors $\boldsymbol{\xi} = \{\xi_\gamma : \gamma \in \mathcal{P}\}$
- Two-contour factors $\boldsymbol{\varphi} = \{\varphi(\gamma, \gamma') : \gamma, \gamma' \in \mathcal{P}\}$

[plus measurability and integrability hypotheses]

Model defined by measures on $\sum_n \mathcal{P}_\Lambda^n$ whith projections

$$\frac{1}{\Xi_\Lambda} \frac{z^n}{n!} \Big[\prod_{1 \le i \le j \le n} \varphi(\gamma_i, \gamma_j) \Big] \Big[\prod_{1 \le i \le n} \xi_{\gamma_i} \, d\gamma_i \Big]$$

$$\Xi_\Lambda(z, \boldsymbol{\xi}, \boldsymbol{\varphi}) \;=\; \sum_{n \ge 0} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} \Big[\prod_{1 \le i \le j \le n} \varphi(\gamma_i, \gamma_j) \Big] \Big[\prod_{1 \le i \le n} \xi_{\gamma_i} \, d\gamma_i \Big]$$

The levels

Following Sokal:

- *Level 1*: $0 \leq |\varphi(\gamma, \gamma')| \leq 1$, for all γ, γ'
  - This corresponds to a *repulsive* gas $[\phi(x_i - x_j) \geq 0]$
  - In this case, $|\Xi_\Lambda| \leq \exp\big(\int_{\mathcal{P}_\Lambda} |z_\gamma| \, d\gamma\big)$
- *Level 2*: $\varphi(\gamma, \gamma) = 0$ (hard-core self repulsion), $0 \leq |\varphi(\gamma, \gamma')| \leq 1$, for all γ, γ'
- *Level 3*: $\varphi(\gamma, \gamma') \in \{0, 1\}$, $\varphi(\gamma, \gamma) = 0$ for all γ, γ' (hard core self and mutual repulsion)

In stat mech: short-range repulsion but long-range *attraction*

Graph-theoretical set-up

It involves *complete graphs*

$$\Xi_\Lambda(\boldsymbol{z},\boldsymbol{\varphi}) \;=\; \sum_{n\geq 0}\frac{1}{n!}\sum_{(x_1,\ldots,x_n)\in\mathcal{P}_\Lambda^n}\Big[\prod_{1\leq i\leq n} z_{x_i}\Big]\Big[\prod_{1\leq i\leq j\leq n}\varphi(x_i,x_j)\Big]$$

If $\varphi(x,x)=0$ for all $x\in\mathcal{P}$ (level 2),

$$\Xi_\Lambda(\boldsymbol{z},\boldsymbol{\varphi}) \;=\; \sum_{\Gamma\subset\mathcal{P}_\Lambda}\Big[\prod_{x\in\Gamma} z_x\Big]\Big[\prod_{\{x,y\}\subset\Gamma}\varphi(x,y)\Big]$$

Occupation numbers

In general, terms depend on *occupation numbers*

$$n_x(x_1, \ldots, x_n) \;=\; \#\{1 \le i \le n : x_i = x\}$$

After a bit of combinatorics (coming soon):

$$\Xi_\Lambda(\boldsymbol{z}, \boldsymbol{\varphi}) \;=\; \sum_{\boldsymbol{n} \ge \boldsymbol{0}} \Bigg[\prod_{x \in \mathcal{P}_\Lambda} \frac{z_x^{n_x}\, \varphi(x,x)^{n_x(n_x-1)/2}}{n_x!} \Bigg] \Bigg[\prod_{\{x,y\} \subset \mathcal{P}_\Lambda} \varphi(x,y)^{n_x n_y} \Bigg]$$

In particular, a system with mutual, but not necessarily self-, exclusion ($\varphi(x,y) = 0, 1$ for $x \neq y$, $0 \le \varphi(x,x) \le 1$) is equivalent to a full hard-core (level 1) with

$$\widetilde{z}_x \;=\; \sum_{n \ge 1} \frac{z_x^n\, \varphi(x,x)^{n(n-1)/2}}{n!}$$

Part IV

Algebraic properties of the expansion

We discuss

- Algebraic properties of the coefficients of the series
- Expressions for ϕ^T

We shall present three approaches:

- Derivation using multivariate formal power series
- Verification (valid also for the continuous case)
- Elegant algebraic approach

Outline

Derivation for countable polymers

Issue: write a series

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \phi(\gamma_1,\ldots,\gamma_n)\; z_{\gamma_1} z_{\gamma_2}\ldots z_{\gamma_n}$$

as a *formal* exponential of another *formal* series in $(z_\gamma)_{\gamma\in\mathcal{P}}$

$$\Xi_\Lambda(\boldsymbol{z}) \;\stackrel{\mathrm{F}}{=}\; \exp\Big\{\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \phi^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\ldots z_{\gamma_n}\Big\}$$

The series between curly brackets is the *cluster expansion*

Multiplicity functions

In general, we are dealing with series of the form

$$F(\boldsymbol{z}) \;=\; \sum_{n\geq 0}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} a(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\cdots z_{\gamma_n}$$

Let us not assume anything about the coefficients other than

$a(\gamma_1,\ldots,\gamma_n)$ is symmetric under permutations of $(\gamma_1,\ldots,\gamma_n)$

Therefore, $a(\gamma_1,\ldots,\gamma_n)$ is a fcn. of the *multiplicty function*:

$$\boldsymbol{M} : \mathcal{P}^{(\mathbb{N})} \longrightarrow \mathbb{N}^{(\mathcal{P})}$$

$$\bigl[\boldsymbol{M}(\gamma_1,\ldots,\gamma_n)\bigr]_\gamma \;=\; \#\bigl\{i : \gamma_i = \gamma\bigr\}$$

Exponential generating functions

Let $a(\boldsymbol{\alpha}) = a(\gamma_1, \ldots, \gamma_n)$ if $\boldsymbol{M}(\gamma_1, \ldots, \gamma_n) = \boldsymbol{\alpha}$. Then

$$F(\boldsymbol{z}) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| = n} a(\boldsymbol{\alpha})\, N_{\boldsymbol{\alpha}}\, \boldsymbol{z}^{\boldsymbol{\alpha}}$$

where $|\boldsymbol{\alpha}| = \sum_\gamma \alpha_\gamma$ and

$$\begin{aligned} N_{\boldsymbol{\alpha}} &= \Big\{ (\gamma_1, \ldots, \gamma_{|\boldsymbol{\alpha}|}) : \boldsymbol{M}(\gamma_1, \ldots, \gamma_{|\boldsymbol{\alpha}|}) = \boldsymbol{\alpha} \Big\} \\ &= \frac{|\boldsymbol{\alpha}|}{\prod_\gamma \alpha_\gamma !} = \frac{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}!} \end{aligned}$$

Then

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}} \frac{a(\boldsymbol{\alpha})}{\boldsymbol{\alpha}!}\, \boldsymbol{z}^{\boldsymbol{\alpha}}$$

Multivariate exponential generating function

The truncated coefficients

The problem

Given functions $a(\boldsymbol{\alpha})$, find functions $a^{\mathrm{T}}(\boldsymbol{\alpha})$ s.t.

$$\sum_{\boldsymbol{\alpha}} \frac{a(\boldsymbol{\alpha})}{\boldsymbol{\alpha}!}\, \boldsymbol{z}^{\boldsymbol{\alpha}} \;=\; \exp\Big\{ \sum_{\boldsymbol{\beta}} \frac{a^{\mathrm{T}}(\boldsymbol{\beta})}{\boldsymbol{\beta}!}\, \boldsymbol{z}^{\boldsymbol{\beta}} \Big\}$$

Then, $a^{\mathrm{T}}(\gamma_1, \ldots, \gamma_n) = a^{\mathrm{T}}\big(\boldsymbol{M}(\gamma_1, \ldots, \gamma_n)\big)$

The key relation

Equating coefficients of $\boldsymbol{z}^{\boldsymbol{\alpha}}$

$$\frac{a(\boldsymbol{\alpha})}{\boldsymbol{\alpha}!} \;=\; \sum_{k\geq 1} \frac{1}{k!} \sum_{\substack{(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k) \\ :\sum \boldsymbol{\beta}_i = \boldsymbol{\alpha}}} \prod_{i=1}^{k} \frac{a^{\mathrm{T}}(\boldsymbol{\beta}_i)}{\boldsymbol{\beta}_i!} \tag{1}$$

Algebraic facts

Key observation 1:

Previous expression *uniquely* determines a^{T}:

$$\begin{array}{lrcl} |\boldsymbol{\alpha}|=1 & a(\gamma) & = & a^{\mathrm{T}}(\gamma) \\ |\boldsymbol{\alpha}|=2 & a(\gamma_1,\gamma_2) & = & a^{\mathrm{T}}(\gamma_1,\gamma_2)+a^{\mathrm{T}}(\gamma_1)\,a^{\mathrm{T}}(\gamma_2) \\ & & = & a^{\mathrm{T}}(\gamma_1,\gamma_2)+a(\gamma_1)\,a(\gamma_2) \\ |\boldsymbol{\alpha}|=n & & \dots & \text{(induction)} \end{array}$$

Key observation 2:

Better to go back to n-tuples

$$a(\gamma_1,\dots,\gamma_n) \;=\; \boldsymbol{\alpha}! \sum_{k\ge 1} \frac{1}{k!} \sum_{\substack{(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_k) \\ :\sum \boldsymbol{\beta}_i=\boldsymbol{\alpha}}} \prod_{i=1}^{k} \frac{a^{\mathrm{T}}(\gamma_{I_i})}{\boldsymbol{\beta}_i!}$$

$\{I_1,\dots,I_k\}$ partition of $\{1,\dots,n\}$ (subseqs.) s.t. $\boldsymbol{\beta}_i=\boldsymbol{M}(\gamma_{I_i})$

Number of partitions

Q: How many partitions $\{I_1, \ldots, I_k\}$ satisfy $\boldsymbol{\beta}_i = \boldsymbol{M}(\gamma_{I_i})$?

Preliminary example: $\alpha_{\gamma_0} = n$ and $\alpha_\gamma = 0$ for $\gamma \neq \gamma_0$

Then $\left(\boldsymbol{\beta}_i\right)_{\gamma_0} = m_i$ and $\left(\boldsymbol{\beta}_i\right)_\gamma = 0$ for $\gamma \neq \gamma_0$ and

$$\#\Big\{\text{partitions } \{I_1, \ldots, I_k\} \text{ with } |I_i| = m_i\Big\} = \binom{n}{m_1 \cdots m_k}$$

More generally: $\alpha_{\gamma_1} = n_1, \ldots, \alpha_{\gamma_\ell} = n_\ell$, otherwise $\alpha_\gamma = 0$

Do the same for each n_i:

$$\#\,\text{partitions} = \binom{n_1}{m_1^1 \cdots m_k^1} \cdots \binom{n_\ell}{m_1^\ell \cdots m_k^\ell} = \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}_1! \cdots \boldsymbol{\beta}_k!}$$

Defining relation

Bottom line: If a and a^{T} are perm.-sym. and satisfy:

$$(*) \quad a(\gamma_1, \ldots, \gamma_n) = \sum_k \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,,n\}}} a^{\mathrm{T}}(\gamma_{I_1}) \cdots a^{\mathrm{T}}(\gamma_{I_k})$$

Then, as formal power series in $\boldsymbol{z}$,

$$1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)} a(\gamma_1, \ldots, \gamma_n)\, z_{\gamma_1} \cdots z_{\gamma_n}$$
$$= \exp\Big\{ \sum_{n \geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)} a^{\mathrm{T}}(\gamma_1, \ldots, \gamma_n)\, z_{\gamma_1} \cdots z_{\gamma_n} \Big\}$$

Inverse relation

The relation $(*)$ can be inverted

$$\begin{aligned} a(\gamma) &= a^T(\gamma) \\ a(\gamma_1, \gamma_2) &= a^T(\gamma_1, \gamma_2) + a^T(\gamma_1)\, a^T(\gamma_2) \end{aligned}$$

Hence

$$\begin{aligned} a^T(\gamma) &= a(\gamma) \\ a^T(\gamma_1, \gamma_2) &= a(\gamma_1, \gamma_2) - a(\gamma_1)\, a(\gamma_2) \end{aligned}$$

Analogously,

$$\begin{aligned} a(\gamma_1, \gamma_2, \gamma_3) &= a^T(\gamma_1, \gamma_2, \gamma_3) + \left[a^T(\gamma_1)\, a^T(\gamma_2, \gamma_3) + 2 \text{ permut.}\right] \\ &\quad + a^T(\gamma_1)\, a^T(\gamma_2)\, a^T(\gamma_3) \end{aligned}$$

thus

$$\begin{aligned} a^T(\gamma_1, \gamma_2, \gamma_3) &= a(\gamma_1, \gamma_2, \gamma_3) - \left[a(\gamma_1)\, a(\gamma_2, \gamma_3) + 2 \text{ permut.}\right] \\ &\quad + 2\, a(\gamma_1)\, a(\gamma_2)\, a(\gamma_3) \end{aligned}$$

Alternative expression

Theorem

The relation (∗) is equivalent to

$$(**) \quad a^{\mathrm{T}}(\gamma_1,\ldots,\gamma_n) \;=\; \sum_{k=1}^{n}(-1)^{k-1}(k-1)! \sum_{\substack{\{I_1,\ldots,I_k\}\\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} a(\gamma_{I_i})$$

Proof: (i) Induction on n,

(ii) Proceed as above but asking

$$\sum_{n\geq 1}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)} a^{\mathrm{T}}(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\cdots z_{\gamma_n} \;=\; \log\Bigl\{1+\sum_{n\geq 1}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)} a(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\cdots z_{\gamma_n}\Bigr\}$$

The "exponential transform"

In fact, the fact the use of labels $1, \ldots, n$ is conventional

Theorem ("Exponential transform")

Let S be a finite set and let $F, G : Parts(S) \longrightarrow \mathbb{C}$. Then,

$$F(A) = \sum_{k} \sum_{\substack{\{B_1,\ldots,B_k\} \\ \text{part. of } A}} \prod_{i=1}^{k} G(B_i) \quad \forall\, A \subset S$$

if and only if

$$G(A) = \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \sum_{\substack{\{B_1,\ldots,B_k\} \\ \text{part. of } A}} \prod_{i=1}^{k} F(B_i) \quad \forall\, A \subset S$$

[c.f. Moebius transform: $F(A) = \sum_{B\subset A} G(B)\ \forall A \subset S \iff G(A) = \sum_{B\subset A} (-1)^{|A\setminus B|} F(B)\ \forall A \subset S$]

In fact, previous expression applies also to the continuous case

Theorem

If

$$(*) \quad a(\gamma_1,\ldots,\gamma_n) = \sum_k \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,,n\}}} a^{\mathrm{T}}(\gamma_{I_1})\cdots a^{\mathrm{T}}(\gamma_{I_k})$$

then, as formal power series in z*,*

$$1 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1,\ldots,\gamma_n)\, \xi_{\gamma_1}\cdots\xi_{\gamma_n}\, d\gamma_1\cdots d\gamma_n$$
$$= \exp\Big\{ \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a^{\mathrm{T}}(\gamma_1,\ldots,\gamma_n)\, \xi_{\gamma_1}\cdots\xi_{\gamma_n}\, d\gamma_1\cdots d\gamma_n \Big\}$$

[This results includes the discrete case!]

First proof

Replace $(*)$ in the original series and use combinatorics:

$$1 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n \;=$$

$$1 + \sum_{n\geq 1} \frac{z^n}{n!} \sum_{k\geq 1} \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} \Big[\int_{\mathcal{P}_\Lambda^{|I_i|}} a^{\mathrm{T}}(\gamma_{I_i})\, \boldsymbol{\xi}^{\gamma_{I_i}}\, d\gamma_{I_i} \Big]$$

The integral over $d\gamma_{I_i}$ depends only on $|I_i| =: \ell_i$

There are

$$\frac{1}{k!} \binom{n}{\ell_1 \cdots \ell_k}$$

ways to choose $\{I_1, \cdots, I_k\}$ with $|I_i| = \ell_i$

First proof (conclusion)

Hence

$$1+\sum_{n\geq 1}\frac{z^n}{n!}\int_{\mathcal{P}_\Lambda^n} a(\gamma_1^n)\,\boldsymbol{\xi}^{\gamma_1^n}\,d\gamma_1^n$$

$$= \quad 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{k\geq 1}\sum_{\substack{(\ell_1,\ldots,\ell_k):\\ \ell_1+\cdots+\ell_k=n}}\frac{n!}{k!}\prod_{i=1}^{k}\Big[\frac{z^{\ell_i}}{\ell_i!}\int_{\mathcal{P}_\Lambda^{\ell_i}} a^{\mathrm{T}}(\gamma_1^{\ell_i})\,\boldsymbol{\xi}^{\gamma_1^{\ell_i}}\,d\gamma_1^{\ell_i}\Big]$$

$$= \quad 1+\sum_{k\geq 1}\frac{1}{k!}\Big[\sum_{\ell\geq 1} z^{\ell_i}\int_{\mathcal{P}_\Lambda^{\ell}} a^{\mathrm{T}}(\gamma_1^{\ell})\,\boldsymbol{\xi}^{\gamma_1^{\ell}}\,d\gamma_1^{\ell}\Big]^k \qquad \square$$

Elegant proof

Two ingredients:

(i) An association

$$\underline{a} = \{a_n : \mathcal{P}^n \to \mathbb{C}\} \quad \longleftrightarrow \quad a_0 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n$$

(ii) An operation "$*$" such that

$$\underline{a} * \underline{b} \quad \longleftrightarrow$$
$$\Big[a_0 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n\Big] \Big[b_0 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} b(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n\Big]$$

Algebraic setup: Basic definitions

(i) In $\underline{A} = \{\underline{a}\}$ let us define the product

$$(\underline{a} * \underline{b})_n(\gamma_1^n) \;:=\; \sum_{\substack{(I_1,I_2)\\ part.\ of\{1,\ldots,n\}}} a_{|I_1|}(\gamma_{I_1})\, b_{|I_2|}(\gamma_{I_2})$$

(ii) For each integrable function $\boldsymbol{\xi} = \{\xi_\gamma : \gamma \in \mathcal{P}\}$ let

$$\langle \boldsymbol{\xi}\,,\, \underline{a}\rangle(z) \;:=\; a_0 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n$$

Algebraic setup: Key calculation

Proposition

For each $\boldsymbol{\xi}$*, the map* $\langle \boldsymbol{\xi} \,,\, \bullet \rangle(z)$ *is a homomorphism from* $(\underline{A}, +, *)$ *to the algebra of formal power series; that is,*

(a) $\langle \boldsymbol{\xi} \,,\, \underline{a} + \underline{b} \rangle(z) \;=\; \langle \boldsymbol{\xi} \,,\, \underline{a} \rangle(z) + \langle \boldsymbol{\xi} \,,\, \underline{b} \rangle(z)$

(b) $\langle \boldsymbol{\xi} \,,\, \underline{a} * \underline{b} \rangle(z) \;=\; \langle \boldsymbol{\xi} \,,\, \underline{a} \rangle(z) \cdot \langle \boldsymbol{\xi} \,,\, \underline{b} \rangle(z)$

Proof: (a) Immediate, (b) exercise (easier than the above check on the exponential. □

The $*$-exponential and $*$-log

$(\underline{A}, +, *)$ is an algebra with unit $\underline{\delta}$ with $(\underline{\delta})_n = \delta n\, 0$

[i.e. $\underline{a} * \underline{\delta} = \underline{a}$ for each $\underline{a} \in \underline{A}$]

Let $\underline{A}_+ = \{\underline{a} \in \underline{A} : a_0 = 0\}$. The series

$$\mathrm{Exp}^*(\underline{b}) \;=\; \underline{\delta} + \underline{b} + \frac{1}{2}\underline{b} * \underline{b} + \frac{1}{3!}\underline{b} * \underline{b} * \underline{b} + \cdots$$

defines a map $\mathrm{Exp}^* : \underline{A} \to \underline{A}_+$

By the same combinatorics as for the usual exp and log series,

$$\mathrm{Log}^*(\underline{a}) \;=\; \underline{a} - \frac{1}{2}\underline{a} * \underline{a} + \frac{1}{3}\underline{a} * \underline{a} * \underline{a} + \cdots$$

$\mathrm{Log}^* : \underline{A}_+ \to \underline{A}$, is the functional inverse of Exp^*:

$$\underline{a} = \mathrm{Exp}^*(\underline{b}) \iff \underline{b} = \mathrm{Log}^*(\underline{a}) \qquad (2)$$

Explicit expressions

In fact, for each argument $(x_1, \ldots, x_n)$ both sums are finite:

$$\left[\mathrm{Exp}^*(\underline{b})\right](x_1^n) \;=\; \sum_k \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} b(\gamma_{I_i}) \;,$$

$$\left[\mathrm{Log}^*(\underline{a})\right](x_1^n) \;=\; \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} a(\gamma_{I_i})$$

and (2) is just a proof of the exponential transform.

Conclusion of the elegant proof

The proof that $(*)$ implies

$$
\begin{aligned}
1 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1,\ldots,\gamma_n)\, \xi_{\gamma_1}\cdots\xi_{\gamma_n}\, d\gamma_1\cdots d\gamma_n \\
= \exp\Big\{ \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a^{\mathrm{T}}(\gamma_1,\ldots,\gamma_n)\, \xi_{\gamma_1}\cdots\xi_{\gamma_n}\, d\gamma_1\cdots d\gamma_n \Big\}
\end{aligned}
$$

reduces then to the statement

As $\langle \boldsymbol{\xi}\,,\, \bullet \rangle(z)$ is an homomorphism,
$\langle \boldsymbol{\xi}\,,\, \mathrm{Exp}^*(\underline{a}^T) \rangle(z) = \exp\big[\langle \boldsymbol{\xi}\,,\, \underline{a}^T \rangle(z)\big]$

Moebius transform reinterpreted

Let $\underline{1} \in \underline{A}$ defined by $1(\gamma_1^n) = 1$ for each n

Then

$$\left[\underline{a} * \underline{1}\right](\gamma_1^n) \;=\; \sum_{I \subset \{1,\ldots,n\}} a(\gamma_I)$$

To invert this we need $\underline{g}$ s.t. $\underline{1} * \underline{g} = \underline{\delta}$, or

$$\sum_{I \subset \{1,\ldots,n\}} g(\gamma_I) \;=\; \delta_{n\,0}$$

By induction:

$$g(\gamma_1^n) \;=\; (-1)^n$$

The relation

$$\underline{b} = \underline{a} * \underline{1} \iff \underline{a} = \underline{b} * \underline{g}$$

is Moebius transform

Most popular case

$$a(\gamma_1, \ldots, \gamma_n) \;=\; \prod_{\{i,j\}} \varphi(\gamma_i, \gamma_j)$$

$[\varphi(\gamma_i, \gamma_j) = \mathrm{e}^{-\beta U(\gamma_1, \gamma_j)}$; $\beta \to \infty$ for "hard-core"]. Writing

$$\varphi(\gamma_i, \gamma_j) \;=\; 1 + \Big(\varphi(\gamma_i, \gamma_j) - 1\Big) \;=\; 1 + \psi(\gamma_i, \gamma_j)$$

We have

$$\begin{aligned} a(\gamma_1, \ldots, \gamma_n) &= \prod_{\{i,j\}} \Big[1 + \psi(\gamma_i, \gamma_j)\Big] \\ &= \sum_{C \subset G_n} \prod_{e \in G} \psi(\gamma_e) \end{aligned}$$

- G_n =complete graph with vertices $\{1, \ldots, n\}$
- Sum over (not necessarily spanning) subgraphs of G_n
- $E(G)$ = edge set of G

Connected graphs and partitions

Decomposing each G into connected components,

$$a(\gamma_1,\ldots,\gamma_n) \;=\; \sum_{k=1}^{n} \sum_{\substack{\{G_1,\ldots,G_k\} \\ \text{conn. part. of } G_n}} \prod_{i=1}^{k} \Big[\prod_{e\in E(G)} \psi(\gamma_e) \Big]$$

[G_i can be a single vertex, $\prod_\emptyset \equiv 1$]

Grouping graphs with same vertex set:

$$a(\gamma_1,\ldots,\gamma_n) \;=\; \sum_{k=1}^{n} \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} \Big[\sum_{\substack{G\subset G_{I_i} \\ \text{conn. span.}}} \prod_{e\in E(G_i)} \psi(\gamma_e) \Big]$$

THE formula

Conclusion: If

$$a(\gamma_1, \ldots, \gamma_n) \;=\; \prod_{\{i,j\}} \varphi(\gamma_i, \gamma_j)$$

then

$$a^{\mathrm{T}}(\gamma_1, \ldots, \gamma_n) \;=\; \sum_{\substack{G \subset G_n \\ \text{conn. span.}}} \; \prod_{e \in E(G)} \psi(\gamma_e)$$

with

$$\psi(\gamma_i, \gamma_j) \;=\; \varphi(\gamma_i, \gamma_j) - 1$$

Truncated functions for hard core

For hard core:

$$\psi(\gamma_i,\gamma_j) \;=\; 1\!\!1_{\{\gamma_i\sim\gamma_j\}} - 1 \;=\; \begin{cases} -1 & \text{if } \gamma_i \nsim \gamma_j \\ 0 & \text{if } \gamma_i \sim \gamma_j \end{cases}$$

Hence: For each n-tuple $(\gamma_1,\ldots,\gamma_n)$ construct the graph

$\mathcal{G}_{(\gamma_1,\ldots,\gamma_n)}$ with $V(\mathcal{G}) = \{1,\ldots,n\}$ and $E(\mathcal{G}) = \big\{\{i.j\} : \gamma_i \nsim \gamma_j\big\}$

Then

$$\phi^T(\gamma_1,\ldots,\gamma_n) \;=\; \begin{cases} 1 & n=1 \\ \displaystyle\sum_{\substack{G\subset\mathcal{G}_{(\gamma_1,\ldots,\gamma_n)} \\ G \text{ conn. spann.}}} (-1)^{|E(G)|} & n\ge 2\,,\ \mathcal{G} \text{ conn.} \\ 0 & n\ge 2\,,\ \mathcal{G} \text{ not c.} \end{cases}$$

This formula involves a huge number of cancellations

Penrose identity

Penrose realized that these cancellations can be optimally handled through what is now known as the property of *partitionability* of the family of connected spanning subgraphs

Theorem

For any connected graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ *there exists a family of spanning trees —the Penrose trees* $\mathcal{T}_{\mathcal{G}}^{\mathrm{Penr}}$ *— such that*

$$\sum_{G \subset \mathcal{G}} (-1)^{|E(G)|} \;=\; (-1)^{|\mathbb{V}|-1} \left|\mathcal{T}_{\mathcal{G}}^{\mathrm{Penr}}\right|$$

Partitionability of subgraphs

Let

- $\mathbb{G} = (\mathbb{U}, \mathbb{E})$ a finite connected graph
- $\mathcal{C}_{\mathbb{G}} = \{$connected spanning subgraphs of $\mathbb{G}\}$
- $\mathcal{T}_{\mathbb{G}} = \{$trees belonging to $\mathcal{C}_{\mathbb{G}}\}$

Partial-order $\mathcal{C}_{\mathbb{G}}$ by bond inclusion:

$$G \leq \widetilde{G} \quad \Longleftrightarrow \quad E(G) \subset E(\widetilde{G})$$

If $G \leq \widetilde{G}$, let

$$[G, \widetilde{G}] \;=\; \{\widehat{G} \in \mathcal{C}_{\mathbb{G}} : G \leq \widehat{G} \leq \widetilde{G}\}$$

Partition schemes

A *partition scheme* for $\mathcal{C}_{\mathbb{G}}$ is a map

$$\begin{aligned} R : \mathcal{T}_{\mathbb{G}} &\longrightarrow \mathcal{C}_{\mathbb{G}} \\ \tau &\longmapsto R(\tau) \end{aligned}$$

such that

(i) $E\big(R(\tau)\big) \supset E(\tau)$, and

(ii) $\mathcal{C}_{\mathbb{G}}$ is the disjoint union of the sets $[\tau, R(\tau)]$, $\tau \in \mathcal{T}_{\mathbb{G}}$.

Penrose scheme

- Fix an enumeration $v_0, v_1, \ldots, v_n$ for the vertices of $\mathbb{G}$
- For each $\tau \in \mathcal{T}_{\mathbb{G}}$ let $d(i)$ = tree distance of v_i to v_0
- $R_{\mathrm{Pen}}(\tau)$ is obtained adding to τ $\{v_i, v_j\} \in \mathbb{E} \setminus E(\tau)$ s.t.

(p1) $d(i) = d(j)$ (edges between vertices of the same generation), or

(p2) $d(i) = d(j) - 1$ and $i < j$ (edges connecting to predecessors with smaller index).

Penrose identity

For a partition scheme R, let

$$\mathcal{T}_R \;:=\; \left\{ \tau \in \mathcal{T}_{\mathbb{G}} \,\middle|\, R(\tau) = \tau \right\}$$

(set of R-*trees*).

Proposition

$$\sum_{G \in \mathcal{C}_{\mathbb{G}}} (-1)^{|E(G)|} \;=\; (-1)^{|\mathbb{V}|-1} \left| \mathcal{T}_R \right|$$

for any partition scheme R

Proof of Penrose identity

For any numbers x_e, $e \in \mathbb{E}$,

$$\sum_{G \in \mathcal{C}_{\mathbb{G}}} \prod_{e \in E(G)} x_e = \sum_{\tau \in \mathcal{T}_{\mathbb{G}}} \prod_{e \in E(\tau)} x_e \sum_{\mathcal{F} \subset E(R(\tau)) \setminus E(\tau)} \prod_{e \in \mathcal{F}} x_e$$

$$= \sum_{\tau \in \mathcal{T}_{\mathbb{G}}} \prod_{e \in E(\tau)} x_e \prod_{e \in E(R(\tau)) \setminus E(\tau)} (1 + x_e)$$

- If $x_e = -1$, the last factor kills the contributions of any tree τ with $E(R(\tau)) \setminus E(\tau) \neq \emptyset$
- For any tree, $\left|E(\tau)\right| = |\mathbb{V}| - 1$

Comments

- Hard-core condition is crucial. If only soft repulsion,

$$|1 + x_e| \leq 1$$

and we get the weaker *tree-graph bound*

$$\Big|\sum_{G \in \mathcal{C}_{\mathbb{G}}} \prod_{e \in E(G)} x_e\Big| \leq \sum_{\tau \in \mathcal{T}_{\mathbb{G}}} \prod_{e \in E(\tau)} |x_e| \;\leq\; |\mathcal{T}_{\mathbb{G}}|$$

- The smaller the number of triangle diagrams, the larger the number of Penrose trees. Hence:

$$\begin{aligned} \mathcal{R}(\mathcal{G}) \quad &\supset \quad \mathcal{R}(\text{tree with larger degrees}) \\ &\supset \quad \mathcal{R}(\text{homogeneous tree with max. degree}) \end{aligned}$$