Cluster expansions: Overview and new convergence results

IV. Algebraic identites, convergence and applications

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The setup

Ingredients

- *Countable* family $\mathcal{P}$ of objects: polymers, animals, ...
- *Incompatibility* constraint: $\gamma \nsim \gamma'$ (with $\gamma \nsim \gamma$)
- *Activities* $\boldsymbol{z} = \{z_\gamma\}_{\gamma \in \mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$.

The basic ("finite-volume") measures

For each *finite* family $\mathcal{P}_\Lambda \subset \mathcal{P}$

$$\mathrm{W}_\Lambda\bigl(\{\gamma_1, \gamma_2, \ldots, \gamma_n\}\bigr) \;=\; \frac{1}{\Xi_\Lambda(\boldsymbol{z})}\, z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{(\gamma_1, \ldots, \gamma_n) \in \mathcal{P}_\Lambda^n} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

Examples: Canonical hard core

Hard-core lattice gas:

- Polymers = vertices of a graph
- Incompatible = neighbors

Every polymer system can be set in this form

Single-call loss networks:

- $\mathcal{P}$ = finite connected families of links of a graph —the *calls*
- z_γ = Poissonian rate for the call γ
- Compatibility = use of disjoint links (disjoint calls)

Examples: Low-T expansions

Ising model at low T:

- Polymers = connected closed surfaces (contours)
- Compatibility = no intersection
- $z_\gamma = \exp\{-2\beta J \, |\gamma|\}$

LTE for Ising ferromagnets:

- $\mathcal{P}$ = connected families of (excited) bonds (contours)
- $z_\gamma \;=\; \exp\big\{-2\beta \sum_{B\in\gamma} J_B\big\}$
- $\gamma \sim \gamma'$ iff $\underline{\gamma} \cap \underline{\gamma}' = \emptyset$ (disjoint bases); ($\underline{\gamma} = \cup\{B : B \in \gamma\}$)

Examples: High-T expansions

General HTE:

- $\mathcal{P} = \{$connected finite subsets of bonds$\}$
- $$z_{\boldsymbol{B}} = \int_{\underline{\boldsymbol{B}}} \prod_{A \in \boldsymbol{B}} (\mathrm{e}^{-\beta\,\phi_A(\omega)} - 1) \bigotimes_{x \in \underline{\boldsymbol{B}}} \mu_E(d\omega_x)$$
- $\boldsymbol{B} \sim \boldsymbol{B}'$ iff $\underline{\boldsymbol{B}} \cap \underline{\boldsymbol{B}}' = \emptyset$ ($\underline{\boldsymbol{B}} = \cup\{B : B \in \boldsymbol{B}\}$)

HTE for Ising ferromagnets:

- $\mathcal{P} = \left\{\boldsymbol{B} \in \mathcal{B}_\Lambda : \underline{\boldsymbol{B}} \text{ connected } , \sum_{B \in \boldsymbol{B}} B = \emptyset\right\}$ (cycles)
- $z_{\boldsymbol{B}} = \prod_{B \in \boldsymbol{B}} \tanh(\beta J_B)$
- $\boldsymbol{B} \sim \boldsymbol{B}'$ iff $\underline{\boldsymbol{B}} \cap \underline{\boldsymbol{B}}' = \emptyset$

Examples: Random geometrical models

FK representation of Potts models:

- $\mathcal{P} = \{\gamma \subset\subset \mathbb{L}\}$
-
$$z_\gamma \;=\; q^{-(|\gamma|-1)} \sum_{\substack{\boldsymbol{B} \subset \mathcal{B}_\gamma \\ (\gamma, \boldsymbol{B}) \text{ connected}}} \; \prod_{\{x,y\} \in \boldsymbol{B}} v_{x\,y}$$
with $v_{x\,y} = \mathrm{e}^{\beta J_{x\,y}} - 1$
- Compatibility = non-intersection
- If $v\{x,y\} = -1 \to$ chromatic polynomial
($\beta \to \infty$ with $J_{x\,y} < 0$, i.e. zero-temperature antiferromagnetic Potts)

Examples: Geometrical polymer models

- $\mathcal{P}$ = family of finite subsets of some set $\mathbb{V}$
- $\gamma \sim \gamma' \iff \gamma \cap \gamma' = \emptyset$

Original polymer models of Gruber and Kunz

Generalizations

Continuous polymers

$$\boldsymbol{z} \longrightarrow z\,\boldsymbol{\xi} \quad , \quad \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \quad \longrightarrow \quad \frac{1}{n!} \int_{\mathcal{P}_\Lambda^n} d\gamma_1 \cdots d\gamma_n$$

$$\Xi_\Lambda(z,\boldsymbol{\xi}) = 1 + \sum_{n\ge 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} \xi_{\gamma_1} \ldots \xi_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}\, d\gamma_1 \cdots d\gamma_n$$

Soft interactions

$$\mathbb{1}_{\{\gamma_j \sim \gamma_k\}} \longrightarrow \varphi(\gamma_j, \gamma_k)$$

Cluster expansions

Write the polynomials (in $(z_\gamma)_{\gamma\in\mathcal{P}}$)

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} z_{\gamma_1} z_{\gamma_2}\ldots z_{\gamma_n}\prod_{j<k}\mathbb{1}_{\{\gamma_j\sim\gamma_k\}}$$

as *formal* exponentials of a *formal* series

$$\Xi_\Lambda(\boldsymbol{z}) \stackrel{\mathrm{F}}{=} \exp\Big\{\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n}\phi^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\ldots z_{\gamma_n}\Big\}$$

- The series between curly brackets is the *cluster expansion*
- $\phi^T(\gamma_1,\ldots,\gamma_n)$: *Ursell* or *truncated* functions (symmetric)
- *Clusters*: Families $\{\gamma_1,\ldots,\gamma_n\}$ s.t. $\phi^T(\gamma_1,\ldots,\gamma_n)\neq 0$
- Clusters are *connected* w.r.t. "$\nsim$"

Classical cluster-expansion strategy

Find a Λ-*independent* polydisc where cluster expansions converge *absolutely*

That is, find $\rho_\gamma > 0$ independent of Λ such that cluster expansions converge absolutely in the region

$$\mathcal{R} \;=\; \Big\{ \boldsymbol{z} : |z_\gamma| \le \rho_\gamma \,,\, \gamma \in \mathcal{P} \Big\}$$

To this, find $\boldsymbol{\rho} > \mathbf{0}$ such that

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;:=\; 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \Big|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\Big| \,\rho_{\gamma_1}\ldots\rho_{\gamma_n}$$

(no Λ!) converges. Within this region

- No Ξ_Λ has a zero
- Explicit series expressions for free energy and correlations
- Explicit ψ-mixing
- Central limit theorem

Part IV

Algebraic properties of the expansion (cont.)

Goals:

- Algebraic properties of the coefficients of the series
- Expressions for ϕ^T

Three approaches:

- Derivation using multivariate formal power series
- Verification (valid also for the continuous case)
- Elegant algebraic approach

Outline

Truncation

The case of measurable polymers

General result
1st proof
Elegant proof
Moebius transform

Correlations

Level-1 case

Penrose identity

Truncated functions for hard core
Penrose identity
Partition schemes
Proof of Penrose identity

Derivation through multiplicity functions

If $a(\gamma_1, \ldots, \gamma_n)$ is symmetric under permutations of $(\gamma_1, \ldots, \gamma_n)$

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{(\gamma_1, \ldots, \gamma_n) \in \mathcal{P}^n} a(\gamma_1, \ldots, \gamma_n)\, z_{\gamma_1} \cdots z_{\gamma_n} \;=\; \sum_{\boldsymbol{\alpha} \geq \mathbf{0}} \frac{a(\boldsymbol{\alpha})}{\boldsymbol{\alpha}!}\, \boldsymbol{z}^{\boldsymbol{\alpha}}$$

where $\boldsymbol{\alpha} = \{\alpha_\gamma : \gamma \in \mathcal{P}\}$, $\alpha_\gamma \in \mathbb{N}$ (multiplicty function)

Hence:

$$\sum_{\boldsymbol{\alpha} \geq \mathbf{0}} \frac{a(\boldsymbol{\alpha})}{\boldsymbol{\alpha}!}\, \boldsymbol{z}^{\boldsymbol{\alpha}} \;=\; \exp\Big\{ \sum_{\boldsymbol{\beta} \geq \mathbf{0}} \frac{a^{\mathrm{T}}(\boldsymbol{\beta})}{\boldsymbol{\beta}!}\, \boldsymbol{z}^{\boldsymbol{\beta}} \Big\}$$

Definitions of truncated coefficients

$$(*) \quad a(\gamma_1, \ldots, \gamma_n) = \sum_k \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} a^{\mathrm{T}}(\gamma_{I_1}) \cdots a^{\mathrm{T}}(\gamma_{I_k})$$

or, equivalently

$$(**) \quad a^{\mathrm{T}}(\gamma_1, \ldots, \gamma_n) = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} a(\gamma_{I_i})$$

The "exponential transform"

In fact, the fact the use of labels $1, \ldots, n$ is conventional

Theorem ("Exponential transform")

Let S be a finite set and let $F, G : Parts(S) \longrightarrow \mathbb{C}$. Then,

$$F(A) = \sum_k \sum_{\substack{\{B_1,\ldots,B_k\} \\ \text{part. of } A}} \prod_{i=1}^{k} G(B_i) \quad \forall\, A \subset S$$

if and only if

$$G(A) = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{\substack{\{B_1,\ldots,B_k\} \\ \text{part. of } A}} \prod_{i=1}^{k} F(B_i) \quad \forall\, A \subset S$$

[c.f. Moebius transform: $F(A) = \sum_{B \subset A} G(B)\ \forall A \subset S \iff G(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} F(B)\ \forall A \subset S$]

General measurable polymers

In fact, previous expression applies also to the continuous case

Theorem

If

$$(*) \quad a(\gamma_1,\ldots,\gamma_n) = \sum_k \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} a^{\mathrm{T}}(\gamma_{I_1})\cdots a^{\mathrm{T}}(\gamma_{I_k})$$

then, as formal power series in z,

$$\begin{aligned} 1 + \sum_{n\ge 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1,\ldots,\gamma_n)\,\xi_{\gamma_1}\cdots\xi_{\gamma_n}\,d\gamma_1\cdots d\gamma_n \\ = \exp\Bigl\{\sum_{n\ge 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a^{\mathrm{T}}(\gamma_1,\ldots,\gamma_n)\,\xi_{\gamma_1}\cdots\xi_{\gamma_n}\,d\gamma_1\cdots d\gamma_n\Bigr\}\end{aligned}$$

[This results includes the discrete case!]

First proof

Replace $(*)$ in the original series and use combinatorics:

$$1 + \sum_{n \geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n \;=$$

$$1 + \sum_{n \geq 1} \frac{z^n}{n!} \sum_{k \geq 1} \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} \Big[\int_{\mathcal{P}_\Lambda^{|I_i|}} a^{\mathrm{T}}(\gamma_{I_i})\, \boldsymbol{\xi}^{\gamma_{I_i}}\, d\gamma_{I_i} \Big]$$

The integral over $d\gamma_{I_i}$ depends only on $|I_i| =: \ell_i$

There are

$$\frac{1}{k!} \binom{n}{\ell_1 \cdots \ell_k}$$

ways to choose $\{I_1, \cdots, I_k\}$ with $|I_i| = \ell_i$

First proof (conclusion)

Hence

$$
\begin{aligned}
&1+\sum_{n\geq 1}\frac{z^n}{n!}\int_{\mathcal{P}_\Lambda^n} a(\gamma_1^n)\,\boldsymbol{\xi}^{\gamma_1^n}\,d\gamma_1^n \\
&= 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{k\geq 1}\sum_{\substack{(\ell_1,\ldots,\ell_k):\\ \ell_1+\cdots+\ell_k=n}}\frac{n!}{k!}\prod_{i=1}^{k}\Big[\frac{z^{\ell_i}}{\ell_i!}\int_{\mathcal{P}_\Lambda^{\ell_i}} a^{\mathrm{T}}(\gamma_1^{\ell_i})\,\boldsymbol{\xi}^{\gamma_1^{\ell_i}}\,d\gamma_1^{\ell_i}\Big] \\
&= 1+\sum_{k\geq 1}\frac{1}{k!}\Big[\sum_{\ell\geq 1} z^{\ell_i}\int_{\mathcal{P}_\Lambda^{\ell}} a^{\mathrm{T}}(\gamma_1^{\ell})\,\boldsymbol{\xi}^{\gamma_1^{\ell}}\,d\gamma_1^{\ell}\Big]^k \qquad \square
\end{aligned}
$$

Elegant proof

Two ingredients:

(i) An association

$$\underline{a} = \{a_n : \mathcal{P}^n \to \mathbb{C}\} \quad \longleftrightarrow \quad a_0 + \sum_{n \geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a_n(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n$$

(ii) An operation "$*$" such that

$$\underline{a} * \underline{b} \quad \longleftrightarrow$$

$$\left[a_0 + \sum_{n \geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a_n(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n\right] \left[b_0 + \sum_{n \geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} b_n(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n\right]$$

Algebraic setup: Basic definitions

(i) In $\underline{A} = \{\underline{a}\}$ let us define the product

$$(\underline{a} * \underline{b})_n(\gamma_1^n) \;:=\; \sum_{\substack{(I_1,I_2) \\ part.\ of \{1,\ldots,n\}}} a_{|I_1|}(\gamma_{I_1})\, b_{|I_2|}(\gamma_{I_2})$$

(ii) For each integrable function $\boldsymbol{\xi} = \{\xi_\gamma : \gamma \in \mathcal{P}\}$ let

$$\langle \boldsymbol{\xi}\,,\, \underline{a}\rangle(z) \;:=\; a_0 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a_n(\gamma_1^n)\, \boldsymbol{\xi}^{\gamma_1^n}\, d\gamma_1^n$$

Algebraic setup: Key calculation

Proposition

For each $\boldsymbol{\xi}$*, the map* $\langle \boldsymbol{\xi}\,, \bullet \rangle(z)$ *is a homomorphism from* $(\underline{A}, +, *)$ *to the algebra of formal power series; that is,*

(a) $\langle \boldsymbol{\xi}\,, \underline{a} + \underline{b} \rangle(z) \;=\; \langle \boldsymbol{\xi}\,, \underline{a} \rangle(z) + \langle \boldsymbol{\xi}\,, \underline{b} \rangle(z)$

(b) $\langle \boldsymbol{\xi}\,, \underline{a} * \underline{b} \rangle(z) \;=\; \langle \boldsymbol{\xi}\,, \underline{a} \rangle(z) \cdot \langle \boldsymbol{\xi}\,, \underline{b} \rangle(z)$

Proof: (a) Immediate, (b) exercise (easier than the above check on the exponential. $\square$

The $*$-exponential and $*$-log

$(\underline{A}, +, *)$ is an algebra with unit $\underline{\delta}$ with $(\underline{\delta})_n = \delta_{n\,0}$

[i.e. $\underline{a} * \underline{\delta} = \underline{a}$ for each $\underline{a} \in \underline{A}$]

Let $\underline{A}_+ = \{\underline{a} \in \underline{A} : a_0 = 0\}$. The series

$$\mathrm{Exp}^*(\underline{b}) \;=\; \underline{\delta} + \underline{b} + \frac{1}{2}\underline{b} * \underline{b} + \frac{1}{3!}\underline{b} * \underline{b} * \underline{b} + \cdots$$

defines a map $\mathrm{Exp}^* : \underline{A} \to \underline{\delta} + \underline{A}_+$

By the same combinatorics as for the usual exp and log series,

$$\mathrm{Log}^*(\underline{a}) \;=\; \underline{a} - \frac{1}{2}\underline{a} * \underline{a} + \frac{1}{3}\underline{a} * \underline{a} * \underline{a} + \cdots$$

$\mathrm{Log}^* : \underline{\delta} + \underline{A}_+ \to \underline{A}$, is the functional inverse of Exp^*:

$$\underline{a} = \mathrm{Exp}^*(\underline{b}) \iff \underline{b} = \mathrm{Log}^*(\underline{a}) \tag{1}$$

Explicit expressions

In fact, for each argument $(x_1, \ldots, x_n)$ both sums are finite:

$$\big[\mathrm{Exp}^*(\underline{b})\big]_n(x_1^n) \;=\; \sum_{k=1}^{n} \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} b_{|I_i|}(\gamma_{I_i}) \;,$$

$$\big[\mathrm{Log}^*(\underline{a})\big]_n(x_1^n) \;=\; \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} a_{|I_i|}(\gamma_{I_i})$$

and (1) is just a proof of the exponential transform.

Conclusion of the elegant proof

The proof that $(*)$ implies

$$
\begin{aligned}
1 + \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a(\gamma_1,\ldots,\gamma_n)\, \xi_{\gamma_1}\cdots\xi_{\gamma_n}\, d\gamma_1\cdots d\gamma_n \\
= \exp\Big\{ \sum_{n\geq 1} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} a^{\mathrm{T}}(\gamma_1,\ldots,\gamma_n)\, \xi_{\gamma_1}\cdots\xi_{\gamma_n}\, d\gamma_1\cdots d\gamma_n \Big\}
\end{aligned}
$$

reduces then to the statement

As $\langle \boldsymbol{\xi}\,,\,\bullet\rangle(z)$ is an homomorphism,
$\langle \boldsymbol{\xi}\,,\,\mathrm{Exp}^*(\underline{a}^T)\rangle(z) = \exp\big[\langle \boldsymbol{\xi}\,,\,\underline{a}^T\rangle(z)\big]$

Moebius transform reinterpreted

Let $\underline{1} \in \underline{A}$ defined by $1_n(\gamma_1^n) = 1$ for each n

Then

$$\left[\underline{a} * \underline{1}\right]_n(\gamma_1^n) \;=\; \sum_{I \subset \{1,\ldots,n\}} a_{|I|}(\gamma_I)$$

To invert this we need $\underline{g}$ s.t. $\underline{1} * \underline{g} = \underline{\delta}$, or

$$\sum_{I \subset \{1,\ldots,n\}} g_{|I|}(\gamma_I) \;=\; \delta_{n\,0}$$

By induction:

$$g_n(\gamma_1^n) \;=\; (-1)^n$$

The relation

$$\underline{b} = \underline{a} * \underline{1} \iff \underline{a} = \underline{b} * \underline{g}$$

is Moebius transform

Correlations: General expression

Let us denote

$$P_\Lambda(\gamma_1,\ldots,\gamma_m) \;=\; \mathrm{Prob}_\Lambda\big(\{\gamma_1,\ldots,\gamma_m\}\big)$$

Then

$$P_\Lambda(\gamma_1,\ldots,\gamma_m) \;=\; \frac{\Xi_\Lambda(\gamma_1,\ldots,\gamma_m)}{\Xi_\Lambda}$$

with

$$\Xi_\Lambda(\gamma_1,\ldots,\gamma_m) \;=\; z_{\gamma_1}\cdots z_{\gamma_m}\sum_{n\geq 0}\frac{z^n}{n!}\int_{\mathcal{P}_\Lambda^n}\phi(\gamma_1^m,\widetilde{\gamma}_1^n)\,\boldsymbol{\xi}^{\widetilde{\gamma}_1^n}\,d\widetilde{\gamma}_1^n$$

Derivation operator

Let us introduce $D_\gamma : \underline{A} \longrightarrow \underline{A}_+$:

$$\left[D_\gamma\, \underline{a}\right]_n(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n) \;=\; a_{n+1}(\gamma, \widetilde{\gamma}_1, \ldots, \gamma_n)$$

More generally, let $D_{\gamma_1^m} = D_{\gamma_m} \cdots D_{\gamma_1}$:

$$\left[D_{\gamma_1^m}\, \underline{a}\right]_n(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n) \;=\; a_{n+m}(\gamma_1^m, \widetilde{\gamma}_1^n)$$

We see that

$$\Xi_\Lambda(\gamma_1^m) \;=\; \langle \boldsymbol{\xi}\,,\, D_{\gamma_1^m} \underline{\phi} \rangle \tag{2}$$

Properties of derivations

The operator D_Γ can be called a derivation because

$$D_\gamma(\underline{a} * \underline{b}) \;=\; D_\gamma(\underline{a}) * \underline{b} \;+\; \underline{a} * D_\gamma(\underline{b})$$

[Proof: exercise]

Hence, using series combinatorics as for the usual exponential

$$D_\gamma\big[\mathrm{Exp}^*(\underline{a})\big] \;=\; D_\gamma(\underline{a}) * \mathrm{Exp}^*(\underline{a})$$

and, more generally,

$$D_{\gamma_1^m}\big[\mathrm{Exp}^*(\underline{a})\big] \;=\; \sum_{k=1}^{m} \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,m\}}} D_{\gamma_{I_1}}(\underline{a}) * \cdots * D_{\gamma_{I_k}}(\underline{a}) * \mathrm{Exp}^*(\underline{a}) \qquad (3)$$

Truncated partitions

From (2)–(3):

$$\Xi_\Lambda(\gamma_1^m) = \left\langle \boldsymbol{\xi}, D_{\gamma_1^m} \mathrm{Exp}^*(\underline{\phi}^T) \right\rangle$$

$$= \sum_{k=1}^{m} \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,m\}}} \left\langle \boldsymbol{\xi}\,,\, D_{\gamma_{I_1}}(\underline{\phi}^T) \right\rangle \cdots \left\langle \boldsymbol{\xi}\,,\, D_{\gamma_{I_k}}(\underline{\phi}^T) \right\rangle$$

$$\times \langle \boldsymbol{\xi}\,,\, \mathrm{Exp}^*(\underline{\phi}^T) \rangle$$

Let us denote

$$\Xi_\Lambda^T(\gamma_1^m) := \left\langle \boldsymbol{\xi}\,,\, D_{\gamma_1^m}(\underline{\phi}^T) \right\rangle$$

$$= z_{\gamma_1} \cdots z_{\gamma_m} \sum_{n \geq 0} \frac{z^n}{n!} \int_{\mathcal{P}_\Lambda^n} \phi^T(\gamma_1^m, \widetilde{\gamma}_1^n)\, \boldsymbol{\xi}^{\widetilde{\gamma}_1^n}\, d\widetilde{\gamma}_1^n$$

[can be estimated through cluster expansion]

Truncated probabilities

Finally,

$$P_\Lambda(\gamma_1,\ldots,\gamma_m) \;=\; \sum_{k=1}^{m} \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,m\}}} \Xi_\Lambda^T(\gamma_{I_1})\cdots\Xi_\Lambda^T(\gamma_{I_k})$$

This allows the control of correlations via cluster expansion

Note that $\underline{P}_\Lambda$ is the exponential transform of $\underline{\Xi}_\Lambda^T$

Hence, by the inversion (log) formula:

$$\Xi_\Lambda^T(\gamma_1^m) \;=\; \sum_{k=1}^{m}(-1)^{k-1}(k-1)! \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,m\}}} P_\Lambda(\gamma_{I_1})\cdots P_\Lambda(\gamma_{I_k})$$

Discrete case

Proposition

As formal power series,

$$\Xi_\Lambda^T(\gamma_1,\ldots,\gamma_m) \;=\; \left(z_{\gamma_1}\frac{\partial}{\partial\gamma_1}\cdots z_{\gamma_m}\frac{\partial}{\partial\gamma_m}\right)\log\Xi$$

Proof. By induction, $m=1$ is enough. Must prove:

Lemma

For symmetric functions $a(\gamma_1,\ldots,\gamma_n)$,

$$\frac{\partial}{\partial\gamma_0}\left(\sum_{n\ge 0}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} a(\gamma_1,\ldots,\gamma_n)\,z_{\gamma_1}\cdots z_{\gamma_n}\right)$$

$$= \sum_{n\ge 0}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} a(\gamma_0,\gamma_1,\ldots,\gamma_n)\,z_{\gamma_1}\cdots z_{\gamma_n}$$

Proof of the lemma

We resort to the identity

$$\sum_{n\geq 0}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} a(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\cdots z_{\gamma_n} \;=\; \sum_{\boldsymbol{\alpha}\geq\mathbf{0}}\frac{a(\boldsymbol{\alpha})}{\boldsymbol{\alpha}!}\,\boldsymbol{z}^{\boldsymbol{\alpha}} \qquad (4)$$

We have

$$\begin{aligned}\frac{\partial}{\partial\gamma_0}\Big(\sum_{n\geq 0}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} a(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\cdots z_{\gamma_n}\Big)\\ &= \sum_{\boldsymbol{\alpha}\geq\mathbf{0}:\alpha_{\gamma_0}\geq 1}\frac{a(\boldsymbol{\alpha})}{(\boldsymbol{\alpha}-\delta_{\gamma_0})!}\,\boldsymbol{z}^{\boldsymbol{\alpha}-\delta_{\gamma_0}}\\ &= \sum_{\boldsymbol{\alpha}\geq\mathbf{0}}\frac{a(\boldsymbol{\alpha}+\delta_{\gamma_0})}{\boldsymbol{\alpha}!}\,\boldsymbol{z}^{\boldsymbol{\alpha}}\end{aligned}$$

which, by (4), is the RHS of the lemma $\square$

Most popular case

$$a(\gamma_1,\ldots,\gamma_n) \;=\; \prod_{\{i,j\}} \varphi(\gamma_i,\gamma_j)$$

$[\varphi(\gamma_i,\gamma_j) = \mathrm{e}^{-\beta\, U(\gamma_1,\gamma_j)}$; $\beta \to \infty$ for "hard-core"]. Writing

$$\varphi(\gamma_i,\gamma_j) \;=\; 1 + \Big(\varphi(\gamma_i,\gamma_j) - 1\Big) \;=\; 1 + \psi(\gamma_i,\gamma_j)$$

We have

$$\begin{aligned} a(\gamma_1,\ldots,\gamma_n) &= \prod_{\{i,j\}} \Big[1 + \psi(\gamma_i,\gamma_j)\Big] \\ &= \sum_{G\subset G_n} \prod_{e\in E(G)} \psi(\gamma_e) \end{aligned}$$

- G_n =complete graph with vertices $\{1,\ldots,n\}$
- Sum over (not necessarily spanning) subgraphs of G_n
- $E(G)$ = edge set of G

Connected graphs and partitions

Decomposing each G into connected components,

$$a(\gamma_1,\ldots,\gamma_n) = \sum_{k=1}^{n} \sum_{\substack{\{G_1,\ldots,G_k\} \\ \text{conn. part. of } G_n}} \prod_{i=1}^{k} \Big[\prod_{e\in E(G)} \psi(\gamma_e) \Big]$$

[G_i can be a single vertex, $\prod_{\emptyset} \equiv 1$]

Grouping graphs with same vertex set:

$$a(\gamma_1,\ldots,\gamma_n) = \sum_{k=1}^{n} \sum_{\substack{\{I_1,\ldots,I_k\} \\ \text{part. of } \{1,\ldots,n\}}} \prod_{i=1}^{k} \Big[\sum_{\substack{G\subset G_{I_i} \\ \text{conn. span.}}} \prod_{e\in E(G_i)} \psi(\gamma_e) \Big]$$

THE formula

Conclusion: If

$$a(\gamma_1, \ldots, \gamma_n) \;=\; \prod_{\{i,j\}} \varphi(\gamma_i, \gamma_j)$$

then

$$a^{\mathrm{T}}(\gamma_1, \ldots, \gamma_n) \;=\; \sum_{\substack{G \subset G_n \\ \text{conn. span.}}} \;\prod_{e \in E(G)} \psi(\gamma_e)$$

with

$$\psi(\gamma_i, \gamma_j) \;=\; \varphi(\gamma_i, \gamma_j) - 1$$

Truncated functions for hard core

For hard core:

$$\psi(\gamma_i,\gamma_j) \;=\; \mathbb{1}_{\{\gamma_i\sim\gamma_j\}} - 1 \;=\; \begin{cases} -1 & \text{if } \gamma_i \nsim \gamma_j \\ 0 & \text{if } \gamma_i \sim \gamma_j \end{cases}$$

Hence: For each n-tuple $(\gamma_1,\ldots,\gamma_n)$ construct the graph

$\mathcal{G}_{(\gamma_1,\ldots,\gamma_n)}$ with $V(\mathcal{G}) = \{1,\ldots,n\}$ and $E(\mathcal{G}) = \big\{\{i.j\} : \gamma_i \nsim \gamma_j\big\}$

Then

$$\phi^T(\gamma_1,\ldots,\gamma_n) \;=\; \begin{cases} 1 & n=1 \\ \displaystyle\sum_{\substack{G\subset\mathcal{G}_{(\gamma_1,\ldots,\gamma_n)}\\ G\ \text{conn. spann.}}} (-1)^{|E(G)|} & n\ge 2\,,\ \mathcal{G}\ \text{conn.} \\ 0 & n\ge 2\,,\ \mathcal{G}\ \text{not c.} \end{cases}$$

This formula involves a huge number of cancellations

Penrose identity

Penrose realized that these cancellations can be optimally handled through what is now known as the property of *partitionability* of the family of connected spanning subgraphs

Theorem

For any connected graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ *there exists a family of spanning trees —the Penrose trees* $\mathcal{T}_{\mathcal{G}}^{\mathrm{Penr}}$*— such that*

$$\sum_{G \subset \mathcal{G}} (-1)^{|E(G)|} \;=\; (-1)^{|\mathbb{V}|-1} \left|\mathcal{T}_{\mathcal{G}}^{\mathrm{Penr}}\right|$$

Partitionability of subgraphs

Let

- $\mathbb{G} = (\mathbb{U}, \mathbb{E})$ a finite connected graph
- $\mathcal{C}_{\mathbb{G}} = \{\text{connected spanning subgraphs of } \mathbb{G}\}$
- $\mathcal{T}_{\mathbb{G}} = \{\text{trees belonging to } \mathcal{C}_{\mathbb{G}}\}$

Partial-order $\mathcal{C}_{\mathbb{G}}$ by bond inclusion:

$$G \le \widetilde{G} \quad \Longleftrightarrow \quad E(G) \subset E(\widetilde{G})$$

If $G \le \widetilde{G}$, let

$$[G, \widetilde{G}] \;=\; \{\widehat{G} \in \mathcal{C}_{\mathbb{G}} : G \le \widehat{G} \le \widetilde{G}\}$$

Partition schemes

A *partition scheme* for $\mathcal{C}_{\mathbb{G}}$ is a map

$$\begin{aligned} R : \mathcal{T}_{\mathbb{G}} &\longrightarrow \mathcal{C}_{\mathbb{G}} \\ \tau &\longmapsto R(\tau) \end{aligned}$$

such that

(i) $E\big(R(\tau)\big) \supset E(\tau)$, and

(ii) $\mathcal{C}_{\mathbb{G}}$ is the disjoint union of the sets $[\tau, R(\tau)]$, $\tau \in \mathcal{T}_{\mathbb{G}}$.

Penrose scheme

- Fix an enumeration $v_0, v_1, \ldots, v_n$ for the vertices of $\mathbb{G}$
- For each $\tau \in \mathcal{T}_{\mathbb{G}}$ let $d(i)$ = tree distance of v_i to v_0
- $R_{\mathrm{Pen}}(\tau)$ is obtained adding to τ $\{v_i, v_j\} \in \mathbb{E} \setminus E(\tau)$ s.t.

(p1) $d(i) = d(j)$ (edges between vertices of the same generation), or

(p2) $d(i) = d(j) - 1$ and $i < j$ (edges connecting to predecessors with smaller index).

Penrose identity

For a partition scheme R, let

$$\mathcal{T}_R \;:=\; \left\{\tau \in \mathcal{T}_{\mathbb{G}} \;\middle|\; R(\tau) = \tau\right\}$$

(set of *R-trees*).

Proposition

$$\sum_{G \in \mathcal{C}_{\mathbb{G}}} (-1)^{|E(G)|} \;=\; (-1)^{|\mathbb{V}|-1} \big|\mathcal{T}_R\big|$$

for any partition scheme R

Proof of Penrose identity

For any numbers x_e, $e \in \mathbb{E}$,

$$\sum_{G \in \mathcal{C}_{\mathbb{G}}} \prod_{e \in E(G)} x_e = \sum_{\tau \in \mathcal{T}_{\mathbb{G}}} \prod_{e \in E(\tau)} x_e \sum_{\mathcal{F} \subset E(R(\tau)) \setminus E(\tau)} \prod_{e \in \mathcal{F}} x_e$$

$$= \sum_{\tau \in \mathcal{T}_{\mathbb{G}}} \prod_{e \in E(\tau)} x_e \prod_{e \in E(R(\tau)) \setminus E(\tau)} (1 + x_e)$$

- If $x_e = -1$, the last factor kills the contributions of any tree τ with $E(R(\tau)) \setminus E(\tau) \neq \emptyset$
- For any tree, $\big|E(\tau)\big| = |\mathbb{V}| - 1$ $\square$

Comments

- Hard-core condition is crucial. If only soft repulsion,

$$|1 + x_e| \leq 1$$

and we get the weaker *tree-graph bound*

$$\Big|\sum_{G \in \mathcal{C}_{\mathbb{G}}} \prod_{e \in E(G)} x_e\Big| \;\leq\; \sum_{\tau \in \mathcal{T}_{\mathbb{G}}} \prod_{e \in E(\tau)} |x_e| \;\leq\; |\mathcal{T}_{\mathbb{G}}|$$

- At any rate we have the identity

$$\sum_{G \in \mathcal{C}_{\mathbb{G}}} \prod_{e \in E(G)} x_e = \sum_{\tau \in \mathcal{T}_{\mathbb{G}}} \prod_{e \in E(\tau)} x_e \prod_{e \in E(R(\tau)) \setminus E(\tau)} (1 + x_e)$$

Tree-with-larger-degrees bound

As Penrose conditions involve loops:

The smaller the number of loops,
the easier to satisfy Penrose conditions

Hence, if for an incompatibility graph $\mathcal{G}$,

$$T_{\mathcal{G}} \;=\; \text{homogeneous tree with max. degree of } \mathcal{G}$$

then

$$\left|\mathcal{T}^{\text{Penr}}_{\mathcal{G},n}\right| \;\subset\; \left|\mathcal{T}^{\text{Penr}}_{T_{\mathcal{G}},n}\right|$$

where $\mathcal{T}_{\mathcal{G},n}$ refers to all trees with n vertices

Hence, for the **univariate** case ($z_\gamma = z$, only # of trees counts):

$$\mathcal{R}(\mathcal{G}) \;\supset\; \mathcal{R}(T_{\mathcal{G}})$$

Part V

Convergence criteria for hard-core polymers

We shall review three types of proofs:

- "Classical" (Cammarota, Brydges): defoliation of trees
- Inductive (Kotecký-Preiss, Dobrushin): "no-cluster-expansion"
- Classical revisited (F.-Procacci): trees from root up

We shall compare results for benchmark examples

Outline

Review of formulas

- Truncated functions for hard core
- Penrose identity

Classical convergence criterium

- Classical majorizing series
- Summing "from leaves down"
- Classical criterium

Inductive approach

Classical approach revisited

- New criterion
- Standard form of the criteria

Proof

- The ingredients
- Convergence condition
- Explanation of the different criteria

THE formula

If

$$a(\gamma_1, \dots, \gamma_n) \;=\; \prod_{\{i,j\}} \varphi(\gamma_i, \gamma_j)$$

then

$$a^{\mathrm{T}}(\gamma_1, \dots, \gamma_n) \;=\; \sum_{\substack{G \subset G_n \\ \text{conn. span.}}} \; \prod_{\{i,j\} \in E(G)} \psi(\gamma_i, \gamma_j)$$

with G_n = complete graph on $\{1, \dots, n\}$ and

$$\psi(\gamma_i, \gamma_j) \;=\; \varphi(\gamma_i, \gamma_j) - 1$$

Truncated functions for hard core

For hard core:

$$\psi(\gamma_i,\gamma_j) \;=\; \mathbb{1}_{\{\gamma_i\sim\gamma_j\}} - 1 \;=\; \begin{cases} -1 & \text{if } \gamma_i \nsim \gamma_j \\ 0 & \text{if } \gamma_i \sim \gamma_j \end{cases}$$

Hence: For each n-tuple $(\gamma_1,\ldots,\gamma_n)$ construct the graph

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Then

$$\phi^T(\gamma_1,\ldots,\gamma_n) \;=\; \begin{cases} 1 & n=1 \\ \displaystyle\sum_{\substack{G\subset\mathcal{G}_{(\gamma_1,\ldots,\gamma_n)} \\ G \text{ conn. spann.}}} (-1)^{|E(G)|} & n\ge 2\,,\ \mathcal{G} \text{ conn.} \\ 0 & n\ge 2\,,\ \mathcal{G} \text{ not c.} \end{cases}$$

This formula involves a huge number of cancellations

Penrose identity

Penrose realized that these cancellations can be optimally handled through what is now known as the property of *partitionability* of the family of connected spanning subgraphs

Theorem

For any connected graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ *there exists a family of spanning trees —the Penrose trees* $\mathcal{T}_{\mathcal{G}}^{\mathrm{Penr}}$*— such that*

$$\sum_{G \subset \mathcal{G}} (-1)^{|E(G)|} \;=\; (-1)^{|\mathbb{V}|-1} \left|\mathcal{T}_{\mathcal{G}}^{\mathrm{Penr}}\right|$$

Penrose scheme

- Fix an enumeration $v_0, v_1, \ldots, v_n$ for the vertices of $\mathcal{G}$
- For each $\tau \in \mathcal{T}_\mathcal{G}$ (thought as a tree rooted in v_0), define

$$d(i) \;=\; \text{tree distance of } v_i \text{ to } v_0$$

- Let $R_{\text{Pen}}(\tau)= \tau$ plus all links $\{v_i, v_j\} \in \mathbb{E} \setminus E(\tau)$ s.t.

  (p1) $d(i) = d(j)$ (edges between vertices of the same generation), or

  (p2) $d(i) = d(j) - 1$ and $i < j$ (edges connecting to predecessors with smaller index).

- Then,

$$\tau \in \mathcal{T}_\mathcal{G}^{\text{Penr}} \iff R_{\text{Pen}}(\tau) = \tau$$

Penrose trees

General graph

A Penrose tree for $\mathcal{G}$ is a spanning tree s.t.

(P1) Brothers are not be neighbors in $\mathcal{G}$ and

(P2) A (generalized) nephew-uncle pair is not linked in $\mathcal{G}$ if nephew has larger index

Cluster-expansion graphs

A Penrose tree for $\mathcal{G}_{(\gamma_0,\ldots,\gamma_n)}$ is a spanning tree s.t.

(P1) Brothers are **in**compatible and

(P2) (Generalized) nephews are incompatible with uncles with smaller index

Tree-graph bound

In conclusion:

$$\left|\phi^T(\gamma_0, \gamma_1, \ldots, \gamma_n)\right| = \left|\mathcal{T}^{\mathrm{Pen}}_{\mathcal{G}_{(\gamma_0, \gamma_1, \ldots, \gamma_n)}}\right|$$

Historically, the *only* way Penrose identity was exploited was through the **tree-graph bound**:

$$\left|\phi^T(\gamma_0, \gamma_1, \ldots, \gamma_n)\right| \leq \left|\mathcal{T}_{\mathcal{G}_{(\gamma_0, \gamma_1, \ldots, \gamma_n)}}\right|$$

where $\mathcal{T}_{\mathcal{G}} = \{\text{connected spanning trees of } \mathcal{G}\}$

"Classical" majorizing series

Using the tree-graph bound,

$$\left|\sum_{G\subset\mathbb{G}}(-1)^{|E(G)|}\right| = \left|\mathcal{T}_{\mathbb{G}}^{\text{Penr}}\right| \le \left|\mathcal{T}_{\mathbb{G}}\right|$$

we obtain

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \le \sum_{n\ge 0}\frac{1}{n!}\,\overline{T}_n(\gamma_0)$$

where $\overline{T}_0 = 1$ and

$$\overline{T}_n(\gamma_0) = \sum_{(\gamma_1,\dots,\gamma_n)}\ \sum_{\tau\in\mathcal{T}_{\mathcal{G}_{(\gamma_0,\gamma_1,\dots,\gamma_n)}}} \rho_{\gamma_1}\cdots\rho_{\gamma_n}$$

Contribution of a tree

Interchanging sum over polymers with sum over trees:

$$\overline{T}_n(\gamma_0) = \sum_{\tau \in \mathcal{T}^0_{n+1}} \sum_{\substack{(\gamma_1,\dots,\gamma_n) \text{ s.t.} \\ \tau \subset \mathcal{G}_{(\gamma_0,\gamma_1,\dots,\gamma_n)}}} \rho_{\gamma_1} \cdots \rho_{\gamma_n}$$

$$= \sum_{\tau \in \mathcal{T}^0_{n+1}} \overline{T}_\tau(\gamma_0)$$

where

$$\mathcal{T}^0_{n+1} = \{\text{trees of vertices } 0, 1, \dots n, \text{rooted in } 0\}$$

Geometrical translation-invariant polymers

To compute $\overline{T}_\tau$ start summing over γ's at leaves:

$$\prod_{j=1}^{s_i} \sum_{\gamma_{(i,j)} \not\sim \gamma_i} \rho_{\gamma_{(i,j)}} = \Big[\sum_{\gamma \not\sim \gamma_i} \rho_\gamma \Big]^{s_i}$$

For translation-invariant geometrical polymers,

$$\sum_{\gamma \not\sim \gamma_i} \rho_\gamma \;\leq\; |\gamma_i| \sum_{\gamma \ni 0} \rho_\gamma$$

Then, for each γ_i that is ancestor of leaves

$$\rho_{\gamma_i} \quad \longrightarrow \quad \rho_{\gamma_i} \, |\gamma_i|^{s_i} \Big[\sum_{\gamma \ni 0} \rho_\gamma \Big]^{s_i}$$

Summing "from leaves down"

Iterate! The sum over successive ancestors yields

$$\overline{T}_\tau(\gamma_0) \;\leq\; |\gamma_0| \prod_{i=0}^{n} \Big[\sum_{\gamma \ni 0} \rho_\gamma \; |\gamma|^{s_i} \Big]$$

- This bound depends only on $s_0, s_1, \ldots, s_n$
- The sum over trees τ brings a factor

$$\begin{matrix} \#\text{ trees with coord. nbers} \\ s_0, s_1+1, \ldots, s_n+1 \end{matrix} \;=\; \binom{n}{s_0+1 \; s_1 \; \ldots \; s_n}$$

(Cayley formula)

Classical criterion

In consequence

$$\overline{T}_n(\gamma_0) \;\le\; |\gamma_0|\; n! \sum_{\substack{s_0,s_1,\ldots,s_n \\ \sum s_i = n-1}} \prod_{i=0}^{n} \Big[\sum_{\gamma \ni 0} \rho_\gamma \, \frac{|\gamma|^{s_i}}{s_i!} \Big]$$

Hence

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;\le\; |\gamma_0| \sum_{n \ge 0} \Big[\sum_{\gamma \ni 0} \rho_\gamma \, \mathrm{e}^{|\gamma|} \Big]^n$$

which converges if

$$\sum_{\gamma \ni 0} \rho_\gamma \, \mathrm{e}^{|\gamma|} \;<\; 1$$

[Cammarota (1982), Brydges (1984)]

Inductive arguments

Kotecký-Preiss (1986): Convergence if $a : \mathcal{P} \to [0, \infty)$ s.t.

$$\sum_{\gamma' \not\sim \gamma} \rho_{\gamma'} \, \mathrm{e}^{a(\gamma')} \;\leq\; a(\gamma)$$

Dobrushin (1996): Convergence if $a : \mathcal{P} \to [0, \infty)$ s.t.

$$\rho_\gamma \;\leq\; \Bigl(\mathrm{e}^{a(\gamma)} - 1\Bigr) \, \exp\Bigl\{-\sum_{\gamma' \not\sim \gamma} a(\gamma')\Bigr\}$$

Key: Control $\frac{\Xi_\Lambda}{\Xi_{\Lambda \setminus \{\gamma_0\}}}$ through (deletion-contraction?)

$$\Xi_\Lambda \;=\; \Xi_{\Lambda \setminus \{\gamma_0\}} + z_{\gamma_0} \, \Xi_{\Lambda \setminus \mathcal{N}^*_{\gamma_0}}$$

[$\mathcal{N}^*_{\gamma_0} = \{$polymers incompatible with $\gamma_0\}$]

Dobrushin criterion

Theorem

Assume

$$\rho_\gamma \;\le\; \left(e^{a(\gamma)} - 1\right) \exp\Big\{ - \sum_{\gamma' \not\sim \gamma} a(\gamma') \Big\} \tag{5}$$

Then, if $|z_\gamma| \le \rho_\gamma$

$$\left| \log \left| \frac{\Xi_\Lambda}{\Xi_{\Lambda \setminus \{\gamma_0\}}} \right| \right| \;\le\; a(\gamma_0) \tag{6}$$

Note that if $\Lambda' \subset \Lambda$, telescoping,

$$\left| \log \left| \frac{\Xi_\Lambda}{\Xi_{\Lambda'}} \right| \right| \;\le\; \sum_{\gamma \in \Lambda \setminus \Lambda'} a(\gamma) \;<\; \infty \tag{7}$$

Proof of Dobrushin criterion

By induction on $|\Lambda|$. Start with

$$\left|\frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\right| \;\le\; 1+\rho_{\gamma_0}\left|\frac{\Xi_{\Lambda\setminus\mathcal{N}^*_{\gamma_0}}}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\right|$$

From (7)

$$\left|\frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\right| \;\le\; 1+\rho_{\gamma_0}\,\exp\Bigl\{\sum_{\gamma\not\sim\gamma_0} a(\gamma)\Bigr\}$$

And, by the criterion (5)

$$\left|\frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\right| \;\le\; \mathrm{e}^{a(\gamma_0)}$$

Then use logarithmic inequalities. □

"Standard form" of the criteria

If we substitute

$$\begin{aligned} \mu_\gamma &= \rho_\gamma\, e^{a_\gamma} \quad \text{(Koteck\'y-Preiss)} \\ \mu_\gamma &= e^{a_\gamma} - 1 \quad \text{(Dobrushin)} \end{aligned}$$

We obtain convergence if there exists $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ such that

$$\begin{aligned} \rho_{\gamma_0}\ \exp\Bigl[\sum_{\gamma \not\sim \gamma_0} \mu_\gamma\Bigr] &\le \mu_{\gamma_0} \quad \text{(Koteck\'y-Preiss)} \\ \rho_{\gamma_0} \prod_{\gamma \not\sim \gamma_0} \bigl(1 + \mu_\gamma\bigr) &\le \mu_{\gamma_0} \quad \text{(Dobrushin)} \end{aligned}$$

Comparison D ↔ KP

D improves KP because

$$\prod_{\gamma \not\sim \gamma_0} \left(1 + \mu_\gamma\right) \;\leq\; \exp\Big[\sum_{\gamma \not\sim \gamma_0} \mu_\gamma\Big]$$

Differences:

- D lacks powers μ_γ^ℓ
- D exact for polymers with only self-exclusion

Observations

- It looks as a hierarchy of approximations
- Dobrushin extracts extra information **Which one?**
- Why the form

$$\rho_{\gamma_0}\,\varphi_{\gamma_0}(\boldsymbol{\mu}) \;\le\; \mu_{\gamma_0}\;? \tag{8}$$

Work with A. Procacci:

- All further information must be in Penrose identity
- Form (8) suggests iteration

New condition (with A. Procacci)

For each $\gamma_0 \in \mathcal{P}$ let

$$\Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n \\ \gamma_0 \nsim \gamma_i \,,\, \gamma_i \sim \gamma_j \,,\, 1\leq i,j\leq n}} \mu_{\gamma_1}\mu_{\gamma_2}\ldots\mu_{\gamma_n}$$

(grand-canonical part. funct. of the $\mathcal{G}$-nbhd of γ_0, *including* γ_0)

Theorem

If for $\boldsymbol{\rho} \in [0,\infty)^{\mathcal{P}}$ *there exists a* $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ *such that*

$$\rho_{\gamma_0}\, \Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) \;\leq\; \mu_{\gamma_0} \;, \quad \forall \gamma_0 \in \mathcal{P} \;,$$

then $\boldsymbol{\Pi}(\boldsymbol{\rho})$ *converges for such* $\boldsymbol{\rho}$

Comparison New $\leftrightarrow$ D

The improvement is expressed by the inequality

$$\Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) \;\leq\; \prod_{\gamma \not\sim \gamma_0} \bigl(1 + \mu_\gamma\bigr)$$

LHS contains only monomials of *mutually compatible* polymers

Sources of improvement:

(I1) $\Xi_{\mathcal{N}^*_{\gamma_0}}$ has no triangle diagram (i.e. pairs of neighbors of γ_0 that are themselves neighbors)

(I2) In $\Xi_{\mathcal{N}^*_{\gamma_0}}$, the only monomial containing μ_{γ_0} is μ_{γ_0} itself, (γ_0 is incompatible with all other polymers in $\mathcal{N}^*_{\gamma_0}$)

Intermediate criterium

Our criterium does not have a product form

(Sokal) It may be useful to use the bound

$$\begin{aligned} \Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) &= \mu_{\gamma_0} + \Xi_{\mathcal{N}_{\gamma_0}}(\boldsymbol{\mu}) \\ &\leq \mu_{\gamma_0} + \prod_{\substack{\gamma \nsim \gamma_0 \\ \gamma \neq \gamma_0}} (1 + \mu_\gamma) \end{aligned}$$

to obtain the **Improved Dobrushin criterium**

$$\rho_{\gamma_0} \Big[\mu_{\gamma_0} + \prod_{\substack{\gamma \nsim \gamma_0 \\ \gamma \neq \gamma_0}} (1 + \mu_\gamma) \Big] \leq \mu_{\gamma_0}$$

Summary of conditions

Available convergence conditions are of the form

$$\rho_{\gamma_0}\,\varphi_{\gamma_0}(\boldsymbol{\mu}) \;\le\; \mu_{\gamma_0}$$

with

$$\varphi_{\gamma_0}(\boldsymbol{\mu}) \;=\; \begin{cases} \exp\Big[\sum_{\gamma\in\mathcal{N}^*_{\gamma_0}} \mu_\gamma\Big] & \text{(Kotecký-Preiss)} \\ \prod_{\gamma\in\mathcal{N}^*_{\gamma_0}} \big(1+\mu_\gamma\big) & \text{(Dobrushin)} \\ \mu_{\gamma_0} + \prod_{\gamma\in\mathcal{N}_{\gamma_0}} \big(1+\mu_\gamma\big) & \text{(improved Dobrushin)} \\ \Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) & \text{(new)} \end{cases}$$

Proof. 1st ingredient: Improved tree bound

Retain only (P1): Brothers may not be linked in $\mathcal{G}$

If $\{i, i_1\}$ and $\{i, i_2\}$ are edges of τ, then $\gamma_{i_1} \sim \gamma_{i_2}$

In this way $\boldsymbol{\rho}\,\boldsymbol{\Pi}(\boldsymbol{\rho}) \le \boldsymbol{\rho}^*$, with

$$\rho^*_{\gamma_0} := \rho_{\gamma_0}\Big[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \sum_{\tau\in\mathcal{T}_n^0} \prod_{i=0}^{n} c_{s_i}(\gamma_i, \gamma_{i_1}, \ldots, \gamma_{i_{s_i}})\rho_{\gamma_{i_1}} \ldots \rho_{\gamma_{i_{s_i}}}\Big]$$

where $i_1, \ldots, i_{s_i}$ = descendants of i and

$$c_n(\gamma_0, \gamma_1, \ldots, \gamma_n) = \prod_{i=1}^{n} \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}} \prod_{j=1}^{n} \mathbb{1}_{\{\gamma_i \sim \gamma_j\}}$$

2nd ingredient: Iterative generation of trees

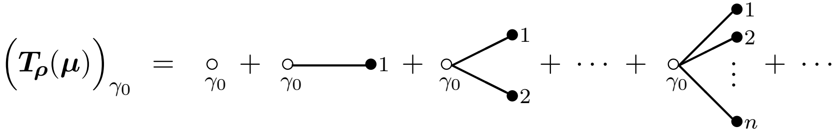
Consider the function $\boldsymbol{T_\rho} : [0,\infty)^{\mathcal{P}} \to [0,\infty]^{\mathcal{P}}$ defined by

$$\Big(\boldsymbol{T_\rho}(\boldsymbol{\mu})\Big)_{\gamma_0} \;=\; \rho_{\gamma_0}\Bigg[1+\sum_{n\ge 1}\frac{1}{n!}\sum_{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n} c_n(\gamma_0,\gamma_1,\dots,\gamma_n)\,\mu_{\gamma_1}\dots\mu_{\gamma_n}\Bigg]$$

or

$$\boldsymbol{T_\rho}(\boldsymbol{\mu}) \;=\; \boldsymbol{\rho}\,\boldsymbol{\varphi}(\boldsymbol{\mu})$$

Diagrammatically:



Summing "from the roots up"

The diagrams of the series

$$\boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\mu})) \;=\; \boldsymbol{T}_{\boldsymbol{\rho}}^{2}(\boldsymbol{\mu})$$

have black dots replaced by each of the preceding diagrams.

That is, $\boldsymbol{T}_{\boldsymbol{\rho}}^{2}(\boldsymbol{\mu}) =$ sums over trees with up to two generations with • in 2nd generation

Likewise, $\boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\mu}) =$ sums over trees with up to n generations with • in n-th generation

Iterating,

$$\boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\rho}) \underset{n\to\infty}{\nearrow} \boldsymbol{\rho}^{*}$$

Alternatively, $\boldsymbol{\rho}^{*}$ generated by replacing $\bullet \;\to\; \boldsymbol{\rho}^{*}$:

$$\boldsymbol{\rho}^{*} \;=\; \boldsymbol{\rho}\,\boldsymbol{\varphi}(\boldsymbol{\rho}^{*}) \qquad \text{or} \qquad \boldsymbol{\rho}^{*} \;=\; \boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\rho}^{*})$$

Convergence

Cheap way to ensure finiteness: Existence of $\boldsymbol{\mu}$ s.t.

$$\boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\mu}) \;\leq\; \boldsymbol{\mu} \tag{9}$$

Then, by positiveness of the terms:

$$\boldsymbol{\rho}^* \;\leq\; \boldsymbol{T}_{\boldsymbol{\rho}}^n(\boldsymbol{\mu}) \;\leq\; \cdots \;\leq \boldsymbol{T}_{\boldsymbol{\rho}}^2(\boldsymbol{\mu}) \;\leq\; \boldsymbol{\mu}$$

Furthermore, if there is convergence, then (9) holds for $\boldsymbol{\mu} = \boldsymbol{\rho}^*$

Theorem $(^*)$

$\boldsymbol{\rho}^*$ converges iff $\boldsymbol{\rho}\,\boldsymbol{\varphi}(\boldsymbol{\mu}) \leq \boldsymbol{\mu}$ for some $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$

Within the region of convergence

(i) $\boldsymbol{T}_{\boldsymbol{\rho}}^n(\boldsymbol{\rho}) \underset{n\to\infty}{\nearrow} \boldsymbol{\rho}^*$

(ii) $\boldsymbol{\rho}^* = \boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\rho}^*)$ or $\boldsymbol{\rho} = \boldsymbol{\rho}^*/\boldsymbol{\varphi}(\boldsymbol{\rho}^*)$:

$$\boldsymbol{\rho}^* = \boldsymbol{f}(\boldsymbol{\rho}) \quad\Longrightarrow\quad \boldsymbol{f}^{-1}(\boldsymbol{\rho}^*) = \frac{\boldsymbol{\rho}^*}{\boldsymbol{\varphi}(\boldsymbol{\rho}^*)}$$

(iii) For each $n \in \mathbb{N}$,

$$\boldsymbol{\rho}\,\boldsymbol{\Pi} \;\leq\; \boldsymbol{\rho}^* \;\leq\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n+1}(\boldsymbol{\mu}) \;\leq\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\mu}) \;\leq\; \boldsymbol{\mu}$$

T_ρ for the new criterion

If

$$c_n(\gamma_0, \gamma_1, \ldots, \gamma_n) = \prod_{i=1}^{n} \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}} \prod_{j=1}^{n} \mathbb{1}_{\{\gamma_i \sim \gamma_j\}}$$

then

$$\Big(\boldsymbol{T_\rho}(\boldsymbol{\mu})\Big)_{\gamma_0} = \rho_{\gamma_0} \Bigg[1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{(\gamma_1, \ldots, \gamma_n) \in \mathcal{P}^n \\ \gamma_0 \nsim \gamma_i \,,\, \gamma_i \sim \gamma_j \,,\, 1 \leq i,j \leq n}} \mu_{\gamma_1} \ldots \mu_{\gamma_n} \Bigg]$$

$$= \rho_{\gamma_0} \, \Xi_{\mathcal{P}_{\gamma_0}}(\boldsymbol{\mu})$$

T_ρ for the Dobrushin criterion

If we replace $\gamma_i \nsim \gamma_j$ by the weaker requirement $\gamma_i \neq \gamma_j$:

$$c_n^{\mathrm{Dob}}(\gamma_0, \gamma_1, \ldots, \gamma_n) \;=\; \prod_{i=1}^{n} \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}} \prod_{j=1}^{n} \mathbb{1}_{\{\gamma_i \neq \gamma_j\}}$$

which yields

$$\begin{aligned}\Big(\boldsymbol{T}_{\boldsymbol{\rho}}^{\mathrm{Dob}}(\boldsymbol{\mu})\Big)_{\gamma_0} &= \rho_{\gamma_0}\Big[1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n \\ \gamma_0 \nsim \gamma_i\,,\,\gamma_i\neq\gamma_j\,,\,1\leq i,j\leq n}} \mu_{\gamma_1}\ldots\mu_{\gamma_n}\Big] \\ &= \rho_{\gamma_0} \prod_{\gamma \nsim \gamma_0} (1+\mu_\gamma)\end{aligned}$$

(Dobrushin condition)

T_ρ for the Kotecký-Preiss criterion

If requirement $\gamma_i \nsim \gamma_j$ is ignored altogether,

$$c_n^{\mathrm{KP}}(\gamma_0, \gamma_1, \ldots, \gamma_n) \;=\; \prod_{i=1}^{n} \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}}$$

and

$$\begin{aligned}\Big(\boldsymbol{T}_{\boldsymbol{\rho}}^{\mathrm{KP}}(\boldsymbol{\mu})\Big)_{\gamma_0} &= \rho_{\gamma_0}\Big[1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n \\ \gamma_0 \nsim \gamma_i\,,\, 1\leq i\leq n}} \mu_{\gamma_1}\ldots\mu_{\gamma_n}\Big] \\ &= \rho_{\gamma_0}\, \exp\Big[\sum_{\gamma\nsim\gamma_0} \mu_\gamma\Big]\end{aligned}$$

(Kotecký-Preiss)

Part VI

Applications and examples

We compare convergence results for

- Incompatibility graphs of bounded degree
- Geometrical polymers
- Zeroes of the chromatic polynomial
- Hard spheres

Outline

Univariate case

Incompatibility graphs of finite degree

Geometrical polymers

Zeroes of chromatic polynomials
- Sources of improvement
- General strategy
- Sokal-Borgs
- Improved bounds

Hard spheres
- The bounds

Perspectives

Univariate case: $z_\gamma = z$

$$\frac{\rho^*}{\rho} \;=\; 1 + \sum_{n=1}^{\infty} \frac{\rho^n}{n!}\Bigg[\sum_{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n} \sum_{\tau\in\mathcal{T}_n^0} \prod_{i=0}^{n} c_{s_i}(\gamma_i,\gamma_{i_1},\dots,\gamma_{i_{s_i}})\Bigg]$$

and

$$\varphi(\mu) \;=\; 1 + \sum_{n\geq 1} \frac{\mu^n}{n!}\Bigg[\sum_{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n} c_n(\gamma_0,\gamma_1,\dots,\gamma_n)\Bigg]$$

Then, the radius of convergence of ρ^* is (exactly!)

$$\sup_{\mu>0} \frac{\mu}{\varphi(\mu)}$$

Single-polymer case

Take $\mathcal{P} = \{\gamma\}$ and $c_{s_i}(\gamma, \gamma, \ldots, \gamma) = c_{s_i}$, then

$$\frac{\rho^*}{\rho} \;=\; 1 + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \Big[\sum_{\tau \in \mathcal{T}_n^0} \prod_{i=0}^{n} c_{s_i} \Big]$$

and

$$\varphi(\mu) \;=\; 1 + \sum_{n \geq 1} c_n \, \frac{\mu^n}{n!}$$

Something known

Particular case: $c_n = 1$

Then,

$$\rho^* = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \rho^n \quad , \quad \varphi(\mu) = \mathrm{e}^{\mu}$$

Theorem (*) implies:

(i) Radius of convergence $= \sup_{\mu>0} \mu\, \mathrm{e}^{-\mu} = \mathrm{e}^{-1}$

(ii) For $0 < x < \mathrm{e}^{-1}$

$$c = f(x) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} x^n \quad \Longleftrightarrow \quad \begin{cases} c = x\, \mathrm{e}^{c} \\ f^{-1}(c) = c\, \mathrm{e}^{-c} \\ f(x) = x\, \mathrm{e}^{f(x)} \end{cases}$$

$f(x) =$ Lambert W function

Comparison: Graphs of maximal degree Δ

Condition	Radius
Kotecký-Preiss	$\dfrac{1}{(\Delta+1)\,e}$
Dobrushin	$\dfrac{\Delta^{\Delta}}{(\Delta+1)^{\Delta+1}}$
Improved Dobrushin =new for $(\Delta-1)$-reg. tree	$\left[1+\dfrac{\Delta^{\Delta}}{(\Delta-1)^{\Delta-1}}\right]^{-1}$
Scott-Sokal	$\dfrac{(\Delta-1)^{(\Delta-1)}}{\Delta^{\Delta}}$ $(*)$
New: $(\Delta+1)$-complete graph	$(\Delta+1)^{-1}$ $(*)$

Explanation: Criteria for graphs of degree Δ

Condition	Criterion
Kotecký-Preiss	$\rho \leq \mu\, e^{-(\Delta+1)\mu}$
Dobrushin	$\rho \leq \dfrac{\mu}{(1+\mu)^{\Delta+1}}$
improved Dobrushin =new for $(\Delta-1)$-reg. tree	$\rho \leq \dfrac{\mu}{\mu+(1+\mu)^{\Delta}}$

Comparison: Graphs of maximal degree 6

Condition	Radius
Kotecký-Preiss	0.052
Dobrushin	0.056
Improved Dobrushin	0.062
Scott-Sokal	0.067
New: Domino in $\mathbb{Z}^2$	0.076
New: Triangular lattice	0.078
New: complete graph	0.142

Explanation: New criteria for graphs of degree 6

Model	Criterion
Domino in $\mathbb{Z}^2$	$\rho \leq \dfrac{\mu}{1+7\mu+9\mu^2}$
Triangular lattice	$\rho \leq \dfrac{\mu}{1+7\mu+8\mu^2+2\mu^3}$
$(\Delta+1)$-complete graph	$\rho \leq \dfrac{\mu}{1+(\Delta+1)\mu}$

Improvements for geometrical polymers

It is useful to pass to functions $a(\gamma)$ defined by $\mu_\gamma = \rho_\gamma \, \mathrm{e}^{a(\gamma)}$

Our new condition becomes

$$1 + \sum_{n\geq 1} \sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\subset\mathcal{P} \\ \gamma_0\cap\gamma_i\neq\emptyset\,,\,\gamma_i\cap\gamma_j=\emptyset\,,\,1\leq i,j\leq n}} \prod_{i=1}^{n} \rho_{\gamma_i} \, \mathrm{e}^{a(\gamma_i)} \;\leq\; \mathrm{e}^{a(\gamma_0)}$$

Keep: each of $\gamma_1,\dots,\gamma_n$ intersects a *different* point in γ_0 (otherwise they would overlap).Hence

(i) $n \leq |\gamma_0|$

(ii) n different points in γ_0 are touched by $\gamma_1 \cup \cdots \cup \gamma_n$

These n points can be chosen in $\binom{|\gamma_0|}{n}$ ways

“New” condition for geometrical polymers

Hence, the left-hand side is less or equal than

$$1 + \sum_{n=1}^{|\gamma_0|} \binom{|\gamma_0|}{n} \Bigg[\sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P} \\ \gamma\ni x}} \rho_\gamma \, \mathrm{e}^{a(\gamma)} \Bigg]^n \;=\; \Bigg[1 + \sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P} \\ \gamma\ni x}} \rho_\gamma \, \mathrm{e}^{a(\gamma)} \Bigg]^{|\gamma_0|}$$

This leads to the condition

$$\sup_{x\in\gamma_0} \sum_{\substack{\gamma\in\mathcal{P} \\ \gamma\ni x}} \rho_\gamma \, \mathrm{e}^{a(\gamma)} \;\le\; \mathrm{e}^{a(\gamma_0)/|\gamma_0|} - 1$$

Gruber-Kunz condition

In practice, $a(\gamma)$ is chosen of the form $a(\gamma) = a\,|\gamma|$, with $a > 0$:

- This the expected optimal asymptotic behavior for $|\gamma|$ large
- Calculations are reduced to the determination of a

[General dependence: to deal better with small polymers]

If, in addition,

$$\sup_{x\in\gamma_0} \quad \longrightarrow \quad \sup_{x\in\mathbb{V}}$$

"new" condition = Gruber-Kunz (1971) condition

Originally proven using Kirkwood-Salzburg, can also be proven inductively

Comparison: Geometrical polymers

Criterion	Condition
Kotecký-Preiss	$\sup_x \sum_{\gamma\in\mathcal{P}:\gamma\ni x} \rho_\gamma \, \mathrm{e}^{a\lvert\gamma\rvert} \;\le\; a$
Dobrushin	$\sup_x \prod_{\gamma\in\mathcal{P}:\gamma\ni x} \Big[1+\rho_\gamma \, \mathrm{e}^{a\lvert\gamma\rvert}\Big] \;\le\; \mathrm{e}^a$
Gruber-Kunz	$\sup_x \sum_{\gamma\in\mathcal{P}:\gamma\ni x} \rho_\gamma \, \mathrm{e}^{a\lvert\gamma\rvert} \;\le\; \mathrm{e}^a - 1$

Zeros of chromatic polynomials

No zeros = convergence of cluster expansion for $\gamma \subset \mathbb{V}$ with

$$z_\gamma(q) \;=\; q^{-(|\gamma|-1)} \sum_{\substack{\mathbf{B}\subset\mathcal{B}_\gamma \\ (\gamma,\boldsymbol{B})\,\text{conn.}}} (-1)^{|\boldsymbol{B}|}$$

Available criteria

$$\sup_x \sum_{\gamma\in\mathcal{P}:\gamma\ni x} \rho_\gamma\, \mathrm{e}^{a|\gamma|} \;\le\; \begin{cases} a & (KP) \\ \mathrm{e}^a - 1 & (GK) \end{cases}$$

Double improvement

Combining above expressions, zeros are excluded if

$$\sum_{n\geq 2} \mathrm{e}^{an}\, C_n^q \;\leq\; \begin{cases} a & (KP) \\ \mathrm{e}^{a}-1 & (GK) \end{cases}$$

with

$$C_n^q \;=\; \sup_{x\in\mathbb{V}} \sum_{\substack{\gamma\subset\mathbb{V}:\ x\in\gamma \\ |\gamma|=n}} \bigl|z_\gamma(q)\bigr|$$

Two sources of improvement:

(i) Use of GK instead of KP

(ii) Better estimation of C_n^q thanks to Penrose

Successive bounds

$$C_n^q \;\le\; \left(\frac{1}{q}\right)^{n-1} T_n$$

with

$$T_n \;=\; \begin{cases} \sup_{v_0 \in \mathbb{V}} \; t_n^{\mathrm{Pen}}(\mathbb{G}, v_0) \\ \sup_{v_0 \in \mathbb{V}} \; t_n(\mathbb{G}, v_0) \\ \dfrac{n^{n-1}}{n!} \, \Delta^{n-1} \end{cases}$$

$t_n(\mathbb{G}, v_0) = \#$ subtrees of $\mathbb{G}$, with n vertices, including v_0

$t_n^{\mathrm{Pen}}(\mathbb{G}, v_0) = \#$ of Penrose subtrees rooted at v_0

General strategy

Chromatic polynomial free of zeros in the region

$$
\begin{aligned}
|q| \quad &\geq \quad \min_{a\geq 0} \, \inf\left\{\kappa : \sum_{n=1}^{\infty} T_n \left[\frac{\mathrm{e}^a}{\kappa}\right]^{n-1} \leq \left\{ \begin{array}{ll} 1 + a\,\mathrm{e}^{-a} & (KP) \\ 2 - \mathrm{e}^{-a} & (GK) \end{array} \right\} \right\} \\
&= \quad \min_{a\geq 0} \, \mathrm{e}^a \left[\sup\left\{ x : F(x) \leq \left\{ \begin{array}{ll} 1 + a\,\mathrm{e}^{-a} & (KP) \\ 2 - \mathrm{e}^{-a} & (GK) \end{array} \right\} \right\}\right]^{-1}
\end{aligned}
$$

with

$$
F(x) \; = \; \sum_{n=1}^{\infty} T_n \, x^{n-1}
$$

Sokal-Borgs bound

For the weakest choice $T_n = n^{n-1}\,\Delta^{n-1}/n!$,

$$F(x) \;=\; \frac{f(\Delta x)}{\Delta x} \;=\; \mathrm{e}^{f(\Delta x)}$$

for f seen above. Hence

$$F(x) \;\leq\; 1 + a\,\mathrm{e}^{-a} \quad\Longrightarrow\quad f(\Delta x) \;\leq\; \ln(1 + a\,\mathrm{e}^{-a})$$

and, as $f^{-1}(c) = c\,\mathrm{e}^{-c}$, there are no zeros if

$$|q| \;\geq\; \min_{a\geq 0} \frac{\exp\Big\{a + \ln(1 + a\,\mathrm{e}^{-a})\Big\}}{\ln(1 + a\,\mathrm{e}^{-a})}\,\Delta$$

GK improvement: $1 + a\,\mathrm{e}^{-a} \;\to\; 2 - \mathrm{e}^{-a}$ $(7.97 \to 6.91)$

Improved bound

$\mathbb{G}$ of maximal degree Δ

Pessimistic estimation:

$$F(x) = \frac{f(x)}{x} \quad \text{with} \quad f(x) = \sum_{n\geq 1} t_n(\Delta)\, x^n$$

$t_n(\Delta) = \#$ of n-vertex subtrees in the Δ-tree incl. a fixed vertex

To construct $f(x)$:

- Start with weight x and choose branches (out of Δ)
- At the end of each branch, repeat!

Hence:

$$f(x) \;=\; x\Big[1 + f(x)\Big]^{\Delta} \quad \text{and} \quad f^{-1}(c) = \frac{c}{(1+c)^{\Delta}}$$

[Exercise: prove this through Theorem (*)]

Sokal bound

$$F(x) \leq 1 + a\,\mathrm{e}^{-a} \implies f(x) \;\leq\; (1 + a\,\mathrm{e}^{-a})^{1/\Delta} - 1$$

$$\implies x \;\leq\; \frac{(1 + a\,\mathrm{e}^{-a})^{1/\Delta} - 1}{1 + a\,\mathrm{e}^{-a}}$$

1st improvement: except for root, only $\Delta - 1$ branches available

$$f_\Delta(x) \;=\; x[1 + f_{\Delta-1}(x)]^\Delta$$

This yields absence of zeros for (Sokal's table)

$$|q| \;\geq\; \min_{a>0} \frac{\mathrm{e}^a(1 + a\mathrm{e}^{-a})^{1-\frac{1}{\Delta}}}{(1 + a\mathrm{e}^{-a})^{\frac{1}{\Delta}} - 1}$$

2nd improvement: $1 + a\mathrm{e}^{-a} \to 2 - \mathrm{e}^{-a}$

Use of Penrose trees

- Penrose trees exclude triangle diagrams
- Root can link to any neighbor
- Other vertices link to neighbors $\neq$ predecessor

For $k = 1, \dots \Delta$, let

$$t_k^{\mathbb{G}} = \sup_{v_0 \in \mathbb{V}} \left| \left\{ U \subset \mathcal{N}_{v_0}^* : \ |U| = k \text{ and } \{v, v'\} \notin \mathbb{E} \, \forall \, v, v' \in U \right\} \right|$$

(maximal number of families of k vertices that have a common neighbor but are not neighbors between themselves)

$$\widetilde{t}_k^{\mathbb{G}} = \sup_{v_0 \in \mathbb{V}} \max_{v \in \mathcal{N}_{v_0}^*} \left| \left\{ U \subset \mathcal{N}_{v_0}^* \setminus \{v\} : \ |U| = k \text{ and } \{v, v'\} \notin \mathbb{E} \, \forall \, v, v' \in U \right\} \right|$$

(same as above but excluding, in addition, one of the neighbors)

Doubly improved bound

Then

$$Z_{\mathbb{G}}(x) = 1 + \sum_{k=1}^{\Delta} t_k^{\mathbb{G}}\, x^k \tag{10}$$

plays the role of $(1+x)^{\Delta}$ in Sokal's argument, and

$$\widetilde{Z}_{\mathbb{G}}(x) = 1 + \sum_{k=1}^{\Delta-1} \widetilde{t}_k^{\mathbb{G}}\, x^k \tag{11}$$

plays the role of $1 + f_{\Delta-1}$. Using also GK:

$$|q| \;\geq\; \min_{a>0} \mathrm{e}^{a}\, \frac{\tilde{Z}_{\mathbb{G}}\Big(Z_{\mathbb{G}}^{-1}(2-\mathrm{e}^{-a})\Big)}{Z_{\mathbb{G}}^{-1}(2-\mathrm{e}^{-a})}$$

Comparison: Zeros of chromatic polynomials

Upper bounds of the radius of the polydisc containing the zeros of the chromatic polynomials for graphs of maximum degree Δ

Δ	General graph		Complete graph	
	Sokal	New	New	Exact
2	13.23	10.72	9.90	2
3	21.14	17.57	15.75	3
4	29.08	24.44	21.58	4
6	44.98	38.24	33.24	6
Any	7.97Δ	6.91Δ	5.83Δ	Δ

Classical bound for the hard-sphere gas

$$\begin{aligned} \varphi_{\gamma_0}(\mu) &= 1 + \sum_{n\geq 1} \frac{\mu^n}{n!} \int_{\Lambda^n} dx_1 \cdots dx_n \prod_i \mathbb{1}_{\{|x_i - x_0| \leq R\}} \\ &= \exp\bigl[V_d(R)\,\mu\bigr] \end{aligned}$$

with $V_d(R)$ = volume of d-dimensional sphere of radius R

Hence convergence if

$$|z|\; V_d(R) \;<\; \max_\mu \frac{\mu}{\exp\bigl[V_d(R)\mu\bigr]} \;=\; \frac{1}{\mathrm{e}}$$

Analycity for the hard-sphere gas: New bound

$$|z|\,V_d(R) \;\le\; \max_{\mu>0} \frac{\mu}{C_d(\mu)}$$

where

$$C_d(\mu) \;=\; \sum_{k\ge 0} \frac{\mu^k}{k!}\,\frac{1}{[V_d(1)]^k} \int_{\substack{|y_i|\le 1\\ |y_i-y_j|>1}} dy_1 \ldots dy_k$$

Hard-sphere gas in two dimensions

If $d = 2$:

$$\text{Classical:} \quad |z|\, V_2(R) \;\leq\; 0.36787\ldots$$

$$\text{New:} \quad |z|\, V_2(R) \;\leq\; 0.5107$$

Directions for further research

- Incorporation of additional constraints in Penrose trees
- Use of other partition schemes
- Inductive proof?
- Extension to polymers with soft interactions (in progress)
- Uncountably many polymers (eg. quantum contours)
- Revisit "classical" results based on cluster expansions

Part VII

Alternative probabilistic scheme

The alternative treatment has the following features:

- It is probabilistic, hence only positive activities
- Basic measures = invariant measures for point processes
- Larger region of validity, but no analyticity
- Yields a "universal" perfect simulation scheme

Outline

The process and its schemes
Basic process
Forward-forward and forward-backwards schemes

Perfect simulation

Probabilistic approach (with P. Ferrari and N. Garcia)

Basic measures are invariant for the following dynamics:

- Attach to each polymer γ a poissonian clock with rate z_γ
- When the clock rings, γ tries to be born
- It succeeds if no other γ' present with $\gamma \nsim \gamma'$
- Once born, the polymer has an $\exp(1)$ lifespan

Alternative scheme

1st step: free process

- Generate first a *free process* where *all* birth are succesful
- Associate to each born polymer γ a space-time *cylinder*

$$C^\gamma = \big(\gamma, [\mathrm{Birth}_{C^\gamma}\,,\,\mathrm{Death}_{C^\gamma}]\big)$$

2nd step: cleaning

To decide whether a given cylinder C^γ remains alive, determine its *clan of ancestors*

$$\begin{aligned}
\boldsymbol{A}_1(C^\gamma) &= \Big\{C' : \mathrm{Base}_{C'} \not\sim \gamma, \mathrm{Birth}_{C^\gamma} \in [\mathrm{Birth}_{C'}\,,\,\mathrm{Death}_{C'}]\Big\} \\
\boldsymbol{A}_{n+1}(C^\gamma) &= \mathbf{A}_1\big(\mathbf{A}_n(C^\gamma)\big) \\
\boldsymbol{A}(C^\gamma) &= \cup_n \boldsymbol{A}_n(C^\gamma)
\end{aligned}$$

Forward-forward scheme

If $\boldsymbol{A}(C^{\gamma})$ is finite. do the cleaning starting from the "mother cylinder"

- Keep mother
- Erase first children
- Keep new mothers
- $\vdots$

This is a *forward-forward* scheme

Backward-forward scheme

Ancestors clan can be constructed backwards
(Poisson and exponential distributions are reversible)

To construct the clan of ancestors of a finite window Λ:

- Generate, backwards, marks at rate $z_\gamma\, \mathrm{e}^{-s}$ for each $\gamma \not\sim \Lambda$
- These are cylinders born at $-s$ and surviving up to 0
- Take the first mark; ignore the rest. If its basis is γ_1
- Repeat with

$$\begin{aligned} \Lambda &\rightarrow \Lambda \cup \{\gamma_1\} \\ s &\rightarrow s - \begin{cases} \mathrm{Birth}_{\gamma_1} & \text{if } \gamma \not\sim \gamma' \\ 0 & \text{if } \gamma \not\sim \Lambda, \gamma \sim \gamma_1 \end{cases} \end{aligned}$$

- $\cdots \longrightarrow \mathbf{A}^\Lambda$

Perfect simulation

If

$$\mathbb{P}(\{\mathbf{A}^{\Lambda} \text{ finite}\}) = 1 \tag{12}$$

cleaning leads *exactly* to a sample of the basic measure

Sufficient conditions for (12)?

- Clan of ancestors defines an *oriented percolation model*
- Lack of percolation $\Longrightarrow$ (12)
- Can dominate by a branching process:
  - branches = ancestors
  - branching rate = mean surface-area of cylinders:

$$\frac{1}{|\gamma|} \sum_{\theta \not\sim \gamma} |\theta| \; z_\theta \; \times \; 1$$

(geometrical case)

Extinction condition

Extinction of the branching process implies (12)

Hence, perfect simulation if

$$\frac{1}{|\gamma|}\sum_{\theta\not\sim\gamma}|\theta|\; z_\theta \;\leq\; 1$$

Under this condition

- $\mathrm{Prob} = \lim_\Lambda \mathrm{Prob}_\Lambda$ exists
- Mixing properties

$$\Big|\mathrm{Prob}(\{\gamma_0,\gamma_1\}) - \mathrm{Prob}(\{\gamma_0\})\,\mathrm{Prob}(\{\gamma_1\})\Big| \;\leq\; \mathrm{e}^{-M\,\mathrm{dist}(\gamma_0,\gamma_1)}$$

- CLT: If A depends on a finite # of polymers

$$\frac{1}{\sqrt{\Lambda}}\sum_{x\in\Lambda}\mathbb{1}_{\{A+x\}} \xrightarrow[\Lambda]{} \mathcal{N}(0,D)$$

with $D = \sum_x \mathrm{Prob}(A\cup A+x)$

Comments

- Perfect simulation of a *finite* window of the *infinite* Prob
- Universal perfect simulation algorithm
- Scheme = alternative definition of Prob
- Hence, new way to prove its properties in a larger region
- No analyticity, no info on zeros of partition functions

-