Fermionic (Grassmann) Representation for Spanning (Hyper)forests and Other Combinatorial Objects

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`cond-mat/0403271`: *PRL* **93**, 080601 (2004)
`arXiv:0706.1509 [math-ph]`: *J. Phys. A* **40**, 13799 (2007)
+ various papers in preparation

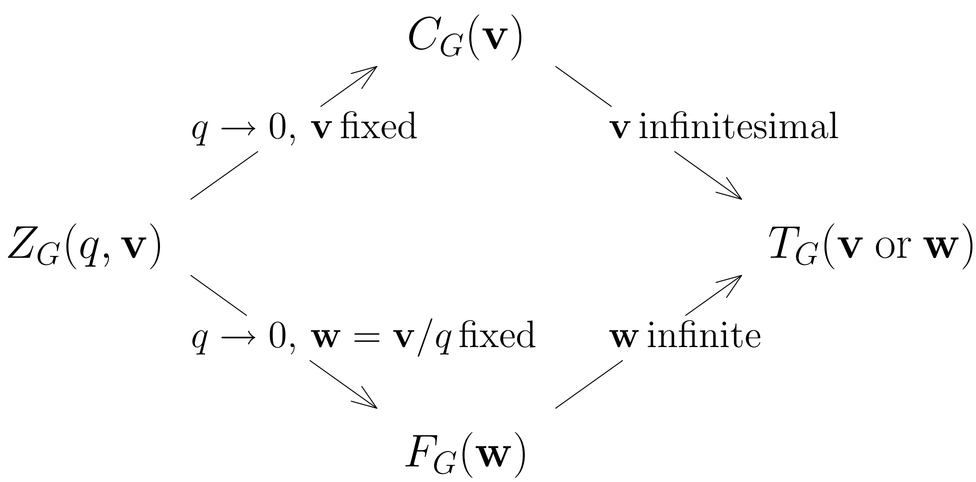
Motivation: $q \to 0$ limits of q-state Potts model

- Finite graph $G = (V, E)$
- Spins $\sigma_i \in \{1, 2, \ldots, q\}$ for sites $i \in V$
- Hamiltonian $H = -\sum_{e=ij \in E} J_e \, \delta(\sigma_i, \sigma_j)$
- Fortuin–Kasteleyn representation:

$$Z_G(q, \mathbf{v}) \;=\; \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e$$

  where $v_e = e^{\beta J_e} - 1$ and $k(A) = \#$ connected components ("clusters") in the graph (V, A)

- $q \to 0$ limits:



where

$$\begin{aligned} C_G(\mathbf{v}) &= \text{gen. poly. of } \textit{connected spanning subgraphs} \\ F_G(\mathbf{w}) &= \text{gen. poly. of } \textit{spanning forests} \\ T_G(\mathbf{v}) &= \text{gen. poly. of } \textit{spanning trees} \end{aligned}$$

Kirchhoff's matrix-tree theorem (1847)

- Laplacian matrix with edge weights $\mathbf{w} = \{w_e\}$:

$$L = L_G(\mathbf{w}) = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum\limits_{k \neq i} w_{ik} & \text{if } i = j \end{cases}$$

- $\det L = 0$ because L has row (and column) sums zero
- Fix a vertex i ("ground") and delete the ith row and column: call this $L_G(\mathbf{w})_{\setminus i}$
- Kirchhoff's matrix-tree theorem:

$$\det L_G(\mathbf{w})_{\setminus i} = T_G(\mathbf{w}) \qquad \text{(independent of } i\text{)}$$

- More generally, delete rows and columns $i_1, \ldots, i_r$:

$$\det L_G(\mathbf{w})_{\setminus \{i_1,\ldots,i_r\}} = \sum_{F \in \mathcal{F}(i_1,\ldots,i_r)} \prod_{e \in F} w_e$$

  (r-component spanning forests with roots $i_1, \ldots, i_r$)

- Formulae also exist for deleting rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$ ("all-minors matrix-tree theorem")

MORAL: Spanning trees and *rooted* spanning forests
$\longrightarrow$ linear algebra!

What about *unrooted* spanning forests, i.e. $F_G(\mathbf{w})$?

Matrix-tree theorem in Grassmann form

- Determinants = *Gaussian* Grassmann integrals
- Introduce Grassmann variables ψ_i, $\bar{\psi}_i$ at each site $i \in V$
- For any matrix A,

$$\int \mathcal{D}(\psi, \bar{\psi})\, e^{\bar{\psi} A \psi} \;=\; \det A$$

and more generally

$$\int \mathcal{D}(\psi, \bar{\psi})\, \bar{\psi}_{i_1} \psi_{i_1} \,\cdots\, \bar{\psi}_{i_r} \psi_{i_r}\, e^{\bar{\psi} A \psi} \;=\; \det A_{\smallsetminus \{i_1, \ldots, i_r\}}$$

- Matrix-tree theorem in Grassmann form:

$$\int \mathcal{D}(\psi, \bar{\psi})\, \bar{\psi}_i \psi_i\, e^{\bar{\psi} L \psi} \;= T_G(\mathbf{w})$$

and more generally

$$\int \mathcal{D}(\psi, \bar{\psi}) \left(\prod_{\alpha=1}^{r} \bar{\psi}_{i_\alpha} \psi_{i_\alpha} \right) e^{\bar{\psi} L \psi} \;=\; \sum_{F \in \mathcal{F}(i_1, \ldots, i_r)} \prod_{e \in F} w_e$$

Might *unrooted* spanning forests $F_G(\mathbf{w})$
$\rightarrow$ *non-Gaussian* Grassmann integral?

ANSWER: Yes!

Spanning forests as non-Gaussian Grassmann integral

Theorem (`cond-mat/0403271`):

$$\int \mathcal{D}(\psi, \bar{\psi}) \, \exp\Big[\bar{\psi} L \psi \, + \, \lambda \sum_i \bar{\psi}_i \psi_i \, - \, \lambda \sum_{ij \in E} w_{ij} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \Big]$$

$$= \, \lambda^{|V|} F_G(\mathbf{w}/\lambda)$$

This model has a hidden OSP(1|2) supersymmetry (see below).

More generally:

- For each connected subgraph $\Gamma = (V_\Gamma, E_\Gamma)$, introduce

$$Q_\Gamma \, = \, \left(\prod_{e \in E_\Gamma} w_e \right) \left(\prod_{i \in V_\Gamma} \bar{\psi}_i \psi_i \right)$$

  (note that each Q_Γ is *even*, hence commutes with everything)

- We prove the identity

$$\int \mathcal{D}(\psi, \bar{\psi}) \, \exp\left[\bar{\psi} L \psi + \sum_\Gamma t_\Gamma Q_\Gamma \right] \, = \sum_{\substack{H \subseteq G \\ H = (H_1, \ldots, H_\ell)}} \left(\prod_{e \in H} w_e \right) \prod_{\alpha=1}^{\ell} W(H_\alpha)$$

  - Sum runs over spanning subgraphs $H \subseteq G$ consisting of connected components $(H_1, \ldots, H_\ell)$
  - Weights $W(H_\alpha)$ are defined by $W(H_\alpha) \, = \, \sum_{\Gamma \prec H_\alpha} t_\Gamma$ where $\Gamma \prec H_\alpha$ means that H_α contains Γ and contains no cycles other than those lying entirely within Γ
  - Proof is *combinatorial*

- Specialize this to

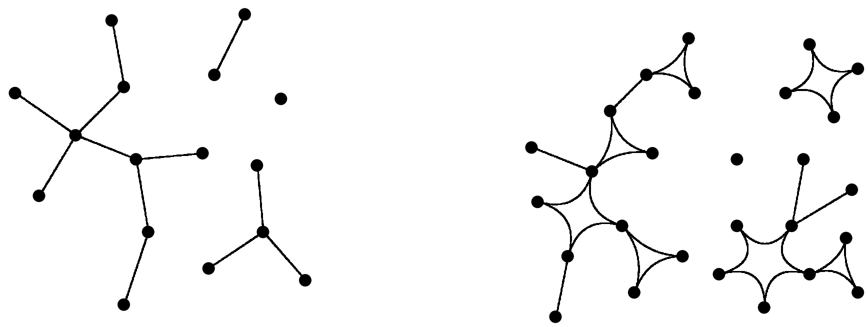
$$
t_\Gamma \;=\; \begin{cases} \lambda & \text{if } \Gamma = \bullet \\ -\lambda & \text{if } \Gamma = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \\ 0 & \text{otherwise} \end{cases}
$$

  and we recover the Theorem for $F_G(\mathbf{w})$

- Other specializations give spanning forests with vertex and edge rootings

- These more general models do *not* have the OSP(1|2) supersymmetry

Generalize from graphs to *hypergraphs*,
forests to *hyperforests* (`arXiv:0706.1509`)



- Potts model with multi-spin interactions
- (More important) Permits simple *algebraic* proofs and insight into role of OSP(1|2) supersymmetry

Potts model on a hypergraph

- Hamiltonian $H = - \sum_{A \in E} J_A \, \delta_A(\sigma)$ where $A = \{i_1, \ldots, i_k\}$ is a hyperedge and

$$\delta_A(\sigma) \;=\; \begin{cases} 1 & \text{if } \sigma_1 = \cdots = \sigma_k \\ 0 & \text{otherwise} \end{cases}$$

- Fortuin–Kasteleyn representation:

$$Z_G(q, \mathbf{v}) \;=\; \sum_{E' \subseteq E} q^{k(E')} \prod_{A \in E'} v_A$$

where $v_A = e^{\beta J_A} - 1$ as before

- One $q \to 0$ limit is gen. poly. of *spanning hyperforests*:

$$F_G(\mathbf{w}) \;=\; \sum_{\substack{E' \subseteq E \\ E' \in \mathcal{F}(G)}} \prod_{A \in E'} w_A$$

A Grassmann subalgebra with unusual properties

- For each subset $A \subseteq V$, associate the monomial $\tau_A = \prod_{i \in A} \bar{\psi}_i \psi_i$
- Note the obvious relations

$$\tau_A \, \tau_B \; = \; \begin{cases} \tau_{A \cup B} & \text{if } A \cap B = \varnothing \\ 0 & \text{if } A \cap B \neq \varnothing \end{cases}$$

- For each subset $A \subseteq V$, define now

$$\begin{aligned} f_A^{(\lambda)} &= \lambda(1 - |A|)\tau_A + \sum_{i \in A} \tau_{A \smallsetminus i} - \sum_{\substack{i,\, j \,\in\, A \\ i \neq j}} \bar{\psi}_i \psi_j \tau_{A \smallsetminus \{i,j\}} \\ &= \left(\lambda(1 - |A|) + \sum_{i,j \in A} \partial_i \bar{\partial}_j \right) \tau_A \end{aligned}$$

  where λ is a real or complex parameter

  (Where does this strange formula come from??)

- We then have the *remarkable relations*

$$f_A^{(\lambda)} \, f_B^{(\lambda)} \; = \; \begin{cases} f_{A \cup B}^{(\lambda)} & \text{if } |A \cap B| = 1 \\ 0 & \text{if } |A \cap B| \geq 2 \end{cases}$$

- **Corollary:** Let $G = (V, E)$ be a *connected* hypergraph. Then

$$\prod_{A \in E} f_A^{(\lambda)} \; = \; \begin{cases} f_V^{(\lambda)} & \text{if } G \text{ is a hypertree} \\ 0 & \text{if } G \text{ is not a hypertree} \end{cases}$$

Spanning *hyper*forests as non-Gaussian Grassmann integral

Theorem (`arXiv:0706.1509`): Let $G = (V, E)$ be a hypergraph, and let $\{w_A\}_{A \in E}$ be hyperedge weights. Then

$$\int \mathcal{D}(\psi, \bar{\psi}) \, \exp\left[\sum_i t_i \bar{\psi}_i \psi_i \; + \sum_{A \in E} w_A f_A^{(\lambda_A)} \right]$$

$$= \sum_{\substack{F \in \mathcal{F}(G) \\ F = (F_1, \ldots, F_\ell)}} \left(\prod_{A \in F} w_A \right) \prod_{\alpha=1}^{\ell} \left(\sum_{i \in V(F_\alpha)} t_i \; - \sum_{A \in E(F_\alpha)} (|A| - 1) \lambda_A \right)$$

where the sum runs over spanning hyperforests F in G with components $F_1, \ldots, F_\ell$, and $V(F_\alpha)$ is the vertex set of the hypertree F_α.

Specializing to $t_i = \lambda$ for all vertices i, we obtain:

Corollary:

$$\int \mathcal{D}(\psi, \bar{\psi}) \, \exp\left[\lambda \sum_i \bar{\psi}_i \psi_i \; + \sum_{A \in E} w_A f_A^{(\lambda)} \right] \; = \sum_{F \in \mathcal{F}(G)} \left(\prod_{A \in F} w_A \right) \lambda^{k(F)}$$

$$= \; \lambda^{|V|} \sum_{F \in \mathcal{F}(G)} \left(\prod_{A \in F} \frac{w_A}{\lambda^{|A|-1}} \right)$$

This latter model has a hidden OSP(1|2) supersymmetry (see below).

Manifestly OSP(1|2)-invariant formulation

- Introduce superfield $\mathbf{n}_i = (\sigma_i, \psi_i, \bar{\psi}_i)$ at each site $i \in V$
- Equip "superspace" $\mathbb{R}^{1|2}$ with the scalar product

$$\mathbf{n}_i \cdot \mathbf{n}_j \;=\; \sigma_i \sigma_j \;+\; \lambda(\bar{\psi}_i \psi_j - \psi_i \bar{\psi}_j)$$

  where $\lambda \neq 0$ is an arbitrary real parameter

- Infinitesimal rotations in $\mathbb{R}^{1|2}$ that leave invariant this scalar product form the Lie superalgebra osp(1|2)
- Now consider σ-model in which the $\mathbf{n}_i$ are constrained to lie on the unit supersphere in $\mathbb{R}^{1|2}$:

$$\mathbf{n}_i \cdot \mathbf{n}_i \;\equiv\; \sigma_i^2 + 2\lambda \bar{\psi}_i \psi_i \;=\; 1$$

  Solve this constraint by writing

$$\sigma_i \;=\; \pm(1 - 2\lambda \bar{\psi}_i \psi_i)^{1/2} \;=\; \pm(1 - \lambda \bar{\psi}_i \psi_i)$$

  (take + sign henceforth)

- We then have

$$f^{(\lambda)}_{\{i,j\}} \;=\; \frac{(\mathbf{n}_i - \mathbf{n}_j)^2}{2\lambda}$$

  and more generally

$$f^{(\lambda)}_{\{i_1,i_2,\ldots,i_k\}} \;=\; \frac{1}{(2\lambda)^{k-1}} \, (\mathbf{n}_{i_1} - \mathbf{n}_{i_2})^2 \, (\mathbf{n}_{i_2} - \mathbf{n}_{i_3})^2 \, \cdots \, (\mathbf{n}_{i_{k-1}} - \mathbf{n}_{i_k})^2$$

  (or use any other tree on the vertex set $\{i_1, \ldots, i_k\}$)

- Finally, the *a priori* measure is

$$\delta(\mathbf{n}_i^2 - 1)\, d\mathbf{n}_i \quad \xrightarrow{\text{integrate out } \sigma_i} \quad e^{\lambda \bar{\psi}_i \psi_i} \, d\psi_i \, d\bar{\psi}_i$$

- **Conclusion:** Generating polynomial for spanning hyperforests in a hypergraph can be written as superfield σ-model:

$$\sum_{F \in \mathcal{F}(G)} \left(\prod_{A \in F} w_A \right) \lambda^{k(F)} \; = \; \int \left(\prod_{i \in V} \delta(\mathbf{n}_i^2 - 1)\, d\mathbf{n}_i \right) \; \times$$

$$\exp \left[\frac{1}{(2\lambda)^{k-1}} \sum_{\{i_1, \ldots, i_k\} \in E} w_{\{i_1, \ldots, i_k\}} (\mathbf{n}_{i_1} - \mathbf{n}_{i_2})^2 \cdots (\mathbf{n}_{i_{k-1}} - \mathbf{n}_{i_k})^2 \right]$$

- **Final remark:** OSP(1|2)-invariant σ-model can also be interpreted as an ordinary $O(N)$-invariant σ-model analytically continued to $N = -1$ ("one fermion-antifermion pair equals -2 bosons"), but at *negative* β. In two dimensions this model is *asymptotically free*.

Work in progress

- Deeper algebraic properties of Grassmann subalgebra generated by $\{f_A^{(\lambda)}\}$
- Extensions to other supersymmetry groups, e.g. OSP$(1|2n)$
- Rigorous nonperturbative formulation of analytic continuation to $N = -1 \;\longrightarrow\;$ Cayley-type identities for powers of determinants